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On moduli of pointed real curves of genus zero

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Abstract. We introduce the moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ of pointed real curves of genus zero and give its natural stratification. The strata of $\mathbb{R}\overline{M}_{(2k,l)}$ correspond to real curves of genus zero with different degeneration types and are encoded by trees with certain decorations. By using this stratification, we calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ and construct the orientation double cover $\mathbb{R}\widetilde{\overline{M}}_{(2k,l)}$ of $\mathbb{R}\overline{M}_{(2k,l)}$.

1. Introduction

The moduli space $\overline{M}_n$ of stable n-pointed (complex) curves of genus zero has been extensively studied as one of the fundamental models of moduli problems in algebraic geometry (see [12, 15, 13, 14, 18]). The moduli space $\overline{M}_n$ carries a set of anti-holomorphic involutions, whose fixed point sets are the moduli spaces of pointed real curves of genus zero. These moduli spaces parameterize the isomorphism classes of pointed curves of genus zero with a real structure. For each of these spaces, a certain set of labeled points stays in the real parts of the curves while other pairs of labeled points are conjugated by the real structures of the curves.

The moduli spaces of pointed real curves have recently attracted attention in various contexts such as multiple ζ-motives [7], representations of quantum groups [4, 9] and Welschinger invariants [22, 23].

The aim of this work is to explore the topological properties of the moduli spaces of pointed real curves of genus zero. Hence, we first introduce a natural combinatorial stratification of the moduli spaces of pointed real curves of genus zero through the stratification of $\overline{M}_n$. Each stratum is determined by the degeneration type of the real curve. They are identified with the product of the spaces of real point configurations in the projective line $\mathbb{CP}^1$ and the moduli spaces $\overline{M}_m$. The degeneration types of the pointed real curves are encoded by trees with corresponding decorations. Secondly, we calculate the first Stiefel-Whitney classes of the moduli spaces in terms of their stratifications. The moduli spaces of pointed real curves are not orientable for $n \geq 5$ and the set of labeled real points of real curves is not empty. We construct the orientation double covers of the moduli spaces for the non-orientable cases. The double covering in this work significantly differs from the 'double covering' in the recent literature on open Gromov-Witten invariants and moduli spaces of pseudoholomorphic discs (see [5, 17]): Our double covering has no boundaries which suits better for the use of intersection theory.

This paper is organized as follows. Section 2 contains a brief overview of facts about the moduli space $\overline{M}_n$. In Section 3, we introduce real structures on $\overline{M}_n$ and real parts $\mathbb{R}\overline{M}_{(2k,l)}$ as moduli spaces of $(2k,l)$-pointed real curves. The following section, the stratification of $\mathbb{R}\overline{M}_{(2k,l)}$ is given according to the degeneration types of pointed real curves of genus zero. In Section 5, the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ is calculated by using the stratification given in Section 4. Then in Section 6, we construct the orientation double coverings $\mathbb{R}\widetilde{M}_{(2k,l)} \to \mathbb{R}\overline{M}_{(2k,l)}$.

In this paper, the genus of the curves is zero except when the contrary is stated explicitly. Therefore, we omit mentioning the genus of the curves.

2. Pointed complex curves and their moduli

This section reviews the basic facts on pointed complex curves of genus zero and their moduli space.

2.1. Pointed curves and their trees

Definition 2.1. An *n-pointed curve* $(\Sigma; \mathbf{p})$ is a connected complex algebraic curve Σ with distinct, smooth, *labeled points* $\mathbf{p} = (p_1, \cdots, p_n) \subset \Sigma$, satisfying the following conditions:

- Σ has only nodal singularities.
- The arithmetic genus of Σ is equal to zero.

A *family of n-pointed curves* over a complex manifold S is a proper, holomorphic map $\pi_S : \mathcal{U}_S \to S$ with n sections $p_1, \cdots, p_n$ such that each geometric fiber $(\Sigma(s); \mathbf{p}(s))$ is an n-pointed curve.

Two such curves, $(\Sigma; \mathbf{p})$ and $(\Sigma'; \mathbf{p}')$, are *isomorphic* if there exists a bi-holomorphic equivalence $\Phi : \Sigma \to \Sigma'$ mapping p_i to p'_i.

An n-pointed curve is *stable* if its automorphism group is trivial (i.e., on each irreducible component, the number of singular points plus the number of labeled points is at least three).

2.1.1. Graphs

Definition 2.2. A *graph* Γ is a collection of finite sets of *vertices* V_Γ and *flags (or half edges)* F_Γ with a boundary map $\partial_\Gamma : F_\Gamma \to V_\Gamma$ and an involution $j_\Gamma : F_\Gamma \to F_\Gamma$ ($j_\Gamma^2 = id$). We call $E_\Gamma = \{(f_1, f_2) \in F_\Gamma^2 \mid f_1 = j_\Gamma f_2 \ \& \ f_1 \neq f_2\}$ the set of *edges*, and $T_\Gamma = \{f \in F_\Gamma \mid f = j_\Gamma f\}$ the set of *tails*. For a vertex $v \in V_\Gamma$, let $F_\Gamma(v) = \partial_\Gamma^{-1}(v)$ and $|v| = |F_\Gamma(v)|$ be the *valency* of v.

We think of a graph Γ in terms of its following *geometric realization* $||\Gamma||$: Consider the disjoint union of closed intervals $\bigsqcup_{f_i \in F_\Gamma} [0,1] \times f_i$ and identify $(0, f_i)$ with $(0, f_j)$ if $\partial_\Gamma f_i = \partial_\Gamma f_j$, and identify (t, f_i) with $(1-t, j_\Gamma f_i)$ for $t \in]0,1[$ and $f_i \neq j_\Gamma f_i$. The geometric realization of Γ has a piecewise linear structure.

Definition 2.3. A *tree* γ is a graph whose geometric realization is connected and simply-connected. If $|v| > 2$ for all $v \in V_\gamma$, then such a tree is called *stable*.

We associate a subtree γ_v for each vertex $v \in V_\gamma$ which is given by $V_{\gamma_v} = \{v\}, F_{\gamma_v} = F_\gamma(v)$, $j_{\gamma_v} = id$, and $\partial_{\gamma_v} = \partial_\gamma$.

Definition 2.4. Let γ and τ be trees with n tails. A *morphism* between these trees $\phi : \gamma \to \tau$ is a pair of maps $\phi_F : F_\tau \to F_\gamma$ and $\phi_V : V_\gamma \to V_\tau$ satisfying the following conditions:

- ϕ_F is injective and ϕ_V is surjective.
- The following diagram commutes

$$\begin{array}{ccc} F_\gamma & \xrightarrow{\partial_\gamma} & V_\gamma \\ {\scriptstyle \phi_F}\uparrow & & \downarrow{\scriptstyle \phi_V} \\ F_\tau & \xrightarrow{\partial_\tau} & V_\tau . \end{array}$$

- $\phi_F \circ j_\tau = j_\gamma \circ \phi_F$.
- $\phi_T := \phi_F|_T$ is a bijection.

An *isomorphism* $\phi : \gamma \to \tau$ is a morphism where ϕ_F and ϕ_V are bijections. We denote the isomorphic trees by $\gamma \approx \tau$.

Each morphism induces a piecewise linear map on geometric realizations.

Lemma 2.1. *Let γ and τ be stable trees with n tails. Any isomorphism $\phi : \gamma \to \tau$ is uniquely defined by its restriction on tails $\phi_T : T_\tau \to T_\gamma$.*

Proof. Let $\phi, \varphi : \gamma \to \tau$ be two isomorphisms such that their restriction on tails are the same. Consider the path ${}_{f_1}P_{f_2}$ in $||\gamma||$ that connects a pair of tails f_1, f_2. The automorphism $\varphi^{-1} \circ \phi$ of γ maps ${}_{f_1}P_{f_2}$ to itself; otherwise, the union of the ${}_{f_1}P_{f_2}$ and its image $\varphi^{-1} \circ \phi({}_{f_1}P_{f_2})$ gives a loop in $||\gamma||$, which contradicts simply-connectedness. Moreover, the restriction of $\varphi^{-1} \circ \phi$ to the path ${}_{f_1}P_{f_2}$ is the identity map since it preserves distances of vertices to tails f_1, f_2. This follows from the compatibility of the automorphism $\varphi^{-1} \circ \phi$ with ∂_γ and j_γ.

The geometric realization $||\gamma||$ of γ can be covered by paths that connects pairs of tails of γ. We conclude that the automorphism $\varphi^{-1} \circ \phi$ is the identity since it is the identity on every such path. □

There are only finitely many isomorphisms classes of stable trees whose set of tails is $T_\gamma = \{1, \cdots, n\}$. We call the isomorphism classes of such trees *n-trees*. We denote the set of all n-trees by $\mathcal{T}ree$.

2.1.2. Dual trees of pointed curves

Let $(\Sigma; \mathbf{p})$ be an n-pointed curve and $\eta : \hat{\Sigma} \to \Sigma$ be its normalization. Let $(\hat{\Sigma}_v; \hat{\mathbf{p}}_v)$ be the following $|v|$-pointed stable curve: $\hat{\Sigma}_v$ is a component of $\hat{\Sigma}$, and $\hat{\mathbf{p}}_v$ is the set of points consisting of the preimages of *special* (i.e. labeled and nodal) points on $\Sigma_v := \eta(\hat{\Sigma}_v)$. The points $\hat{\mathbf{p}}_v = (p_{f_1}, \cdots, p_{f_{|v|}})$ on $\hat{\Sigma}_v$ are ordered by the flags $f_* \in F_\tau(v)$.

Definition 2.5. The *dual tree* γ of an n-pointed curve $(\Sigma;\mathbf{p})$ is the tree consisting of following data:

- V_γ is the set of components of $\hat{\Sigma}$.
- F_γ is the set consisting of the preimages of special points.
- $\partial_\gamma : f \mapsto v$ if and only if $p_f \in \hat{\Sigma}_v$.
- $j_\gamma : f \mapsto f$ if and only if p_f is a labeled point, and $j_\gamma : f_1 \mapsto f_2$ if and only if $p_{f_1} \in \hat{\Sigma}_{v_1}$ and $p_{f_2} \in \hat{\Sigma}_{v_2}$ are the preimages of the nodal point $\Sigma_{v_1} \cap \Sigma_{v_2}$.

Lemma 2.2. *Let Φ be an isomorphism between the n-pointed stable curves $(\Sigma;\mathbf{p})$ and $(\Sigma';\mathbf{p}')$.*
(i) Φ induces an isomorphism ϕ between their dual trees γ,τ.
(ii) Φ is uniquely defined by its restriction on labeled points.

Proof. (i) The result follows from the decomposition of Φ into its restriction to each irreducible component and the Def. 2.4.

(ii) Due to Lemma 2.1, the isomorphism $\phi : \gamma \to \tau$ is determined by the restriction of Φ to the labeled points. The isomorphism ϕ determines which component of Σ is mapped to which component of Σ' as well as the restriction of Φ to the special points. Each component of Σ is rational and has at least three special points. Therefore, the restriction of Φ to a component is uniquely determined by the images of the three special points. □

2.2. Deformations of pointed curves

Let γ be the dual tree of $(\Sigma;\mathbf{p})$ and $\hat{\Sigma} \to \Sigma$ be the normalization. Let $(\hat{\Sigma}_v;\hat{\mathbf{p}}_v)$ be the following $|v|$-pointed stable curve corresponding to the irreducible component Σ_v of $(\Sigma;\mathbf{p})$. Let Ω^1_Σ be the sheaf of Kähler differentials.

The infinitesimal deformations of a nodal curve Σ with divisor $D_{\mathbf{p}} = p_1 + \cdots + p_n$ is canonically identified with the complex vector space

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma), \tag{1}$$

and the obstruction lies in

$$Ext^2_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma).$$

In this case, it is known that there are no obstructions (see, for example [17] or [8]).

The space of infinitesimal deformations is the tangent space of the space of deformations at $(\Sigma;\mathbf{p})$. It can be written explicitly in the following form:

$$\bigoplus_{v\in V_\gamma} H^1(\hat{\Sigma}_v, T_{\hat{\Sigma}_v}(-D_{\hat{\mathbf{p}}_v})) \oplus \bigoplus_{(f_e,f^e)\in E_\gamma} T_{p_{f_e}}\hat{\Sigma} \otimes T_{p_{f^e}}\hat{\Sigma}. \tag{2}$$

The first part corresponds to the equisingular deformations of Σ with the divisor $D_{\hat{\mathbf{p}}_v} = \sum_{f_i\in F_\gamma(v)} p_{f_i}$, and the second part corresponds to the smoothing of nodal points p_e of the edges $e = (f_e, f^e)$ (see [8]).

2.2.1. Combinatorics of degenerations

Let $(\Sigma; \mathbf{p})$ be an n-pointed curve with the dual tree γ. Consider the deformation of a nodal point of $(\Sigma; \mathbf{p})$. Such a deformation of $(\Sigma; \mathbf{p})$ gives a *contraction* of an edge of γ: Let $e = (f_e, f^e) \in E_\gamma$ be the edge corresponding to the nodal point and $\partial_\gamma(e) = \{v_e, v^e\}$, and consider the equivalence relation $\sim$ on the set of vertices, defined by: $v \sim v$ for all $v \in V_\gamma \setminus \{v_e, v^e\}$, and $v_e \sim v^e$. Then, there is an n-tree γ/e whose vertices are $V_\gamma/\sim$ and whose flags are $F_\gamma \setminus \{f_e, f^e\}$. The boundary map and involution of γ/e are the restrictions of ∂_γ and j_γ.

We use the notation $\gamma < \tau$ to indicate that τ is obtained by contracting some edges of γ.

2.3. Stratification of the moduli space $\overline{M}_n$

The moduli space $\overline{M}_n$ is the space of isomorphism classes of n-pointed stable curves. This space is stratified according to degeneration types of n-pointed stable curves which are given by n-trees. The principal stratum M_n corresponds to the one-vertex n-tree and is the quotient of the product $(\mathbb{CP}^1)^n$ minus the diagonals $\Delta = \bigcup_{k<l}\{(p_1, \cdots, p_n) | p_k = p_l\}$ by $Aut(\mathbb{CP}^1) = PSL_2(\mathbb{C})$.

Theorem 2.3 (Knudsen & Keel, [15, 12])**.** *(i) For any $n \geq 3$, $\overline{M}_n$ is a smooth projective algebraic variety of (real) dimension $2n - 6$.*

(ii) Any family of n-pointed stable curves over S is induced by a unique morphism $\kappa : S \to \overline{M}_n$. The universal family of curves $\overline{\mathcal{U}}_n$ of $\overline{M}_n$ is isomorphic to $\overline{M}_{n+1}$.

(iii) For any n-tree γ, there exists a quasi-projective subvariety $D_\gamma \subset \overline{M}_n$ parameterizing the curves whose dual tree is given by γ. D_γ is isomorphic to $\prod_{v \in V_\gamma} M_{|v|}$. Its (real) codimension is $2|E_\gamma|$.

(iv) $\overline{M}_n$ is stratified by pairwise disjoint subvarieties D_γ. The closure of any stratum D_γ is stratified by $\{D_{\gamma'} \mid \gamma' \leq \gamma\}$.

2.3.1. Examples

(i) For $n < 3$, $\overline{M}_n$ is empty due to the definition of n-pointed stable curves. $\overline{M}_3$ is simply a point, and its universal curve $\overline{\mathcal{U}}_3$ is $\mathbb{CP}^1$ endowed with three points.

(ii) The moduli space $\overline{M}_4$ is $\mathbb{CP}^1$ with three points. These points D_{τ_1}, D_{τ_2} and D_{τ_3} correspond to the curves with two irreducible components, and M_4 is the complement of these three points (see Fig. 1). The universal family $\overline{\mathcal{U}}_4$ is a del Pezzo surface of degree five which is obtained by blowing up three points of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

(iii) The moduli space $\overline{M}_5$ is isomorphic to $\overline{\mathcal{U}}_4$. It has ten divisors and each of these divisors contains three codimension two strata. The corresponding 5-trees are shown in Fig. 1.

2.4. Forgetful morphism

We say that $(\Sigma; p_1, \cdots, p_{n-1})$ is obtained by forgetting the labeled point p_n of the n-pointed stable curve $(\Sigma; p_1, \cdots, p_n)$. However, the resulting pointed curve may well

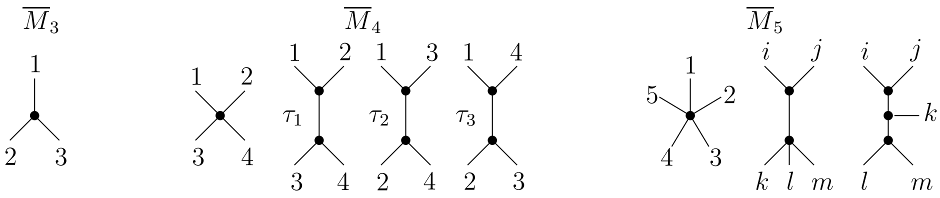

FIGURE 1. Dual trees encoding the strata of $\overline{M}_n$ for $n = 3, 4$, and 5.

be unstable. This happens when the component Σ_v of Σ supporting p_n has only two additional special points. In this case, we contract this component to its intersection point(s) with the components adjacent to Σ_v. With this *stabilization* we extend this map to whole space and obtain $\pi_n : \overline{M}_n \to \overline{M}_{n-1}$. There exists a canonical isomorphism $\overline{M}_n \to \overline{\mathcal{U}}_{n-1}$ commuting with the projections to $\overline{M}_{n-1}$. In other words, $\pi_n : \overline{M}_n \to \overline{M}_{n-1}$ can be identified with the universal family of curves. For details, see [12, 15].

2.5. Automorphisms of $\overline{M}_n$

The open stratum M_n of the moduli space $\overline{M}_n$ can be identified with the orbit space $((\mathbb{CP}^1)^n \setminus \Delta)/PSL_2(\mathbb{C})$. The latter orbit space may be viewed as the configuration space of $(n-3)$ ordered distinct points of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

$$M_n \cong \{\mathbf{p} = (z_1, \cdots, z_n) \in \mathbb{C}^{n-3} \mid z_i \neq z_j \ \forall i \neq j, \ z_{n-2} = 0, z_{n-1} = 1, z_n = \infty\},$$

where $z_i := [z_i : 1]$ are the coordinates of labeled points p_i in an affine chart of $\mathbb{CP}^1$.

Let $\psi = (\psi_1, \ldots, \psi_{n-3}) : M_n \to M_n$ be a non-constant holomorphic map. In [10], Kaliman discovered the following fact:

Theorem 2.4 (Kaliman, [10]). *For $n \geq 4$, every non-constant holomorphic endomorphism $\psi = (\psi_1, \cdots, \psi_{n-3})$ of M_n is an automorphism and its components ψ_r are of the form*

$$\psi_r(\mathbf{p}) = \frac{z_{\varrho(r)} - z_{\varrho(n-2)}}{z_{\varrho(r)} - z_{\varrho(n)}} \Bigg/ \frac{z_{\varrho(n-2)} - z_{\varrho(n-1)}}{z_{\varrho(n)} - z_{\varrho(n-1)}}, \ 1 \leq r \leq n-3$$

where $\varrho \in S_n$ is a permutation not depending on r.

Kaliman's theorem implies the following corollary.

Theorem 2.5 (Kaliman & Lin, [10, 16]). *Every holomorphic automorphism of M_n is produced by a certain permutation $\varrho \in S_n$. Hence, $Aut(M_n) \cong S_n$.*

On the other hand, the permutation group S_n acts on the compactification $\overline{M}_n$ of M_n via relabeling: for $\varrho \in S_n$, there is a holomorphic map ψ_ϱ which is given by

$$\psi_\varrho : (\Sigma; \mathbf{p}) \mapsto (\Sigma; \varrho(\mathbf{p})) := (\Sigma; p_{\varrho(1)}, \cdots, p_{\varrho(n)}). \tag{3}$$

Therefore, the permutation action given in (3) forms a subgroup of holomorphic automorphisms $Aut(\overline{M}_n)$.

Let $Aut_\sharp(\overline{M}_n)$ be the group of holomorphic automorphisms of $\overline{M}_n$ that respect the stratification: $\psi \in Aut_\sharp(\overline{M}_n)$ maps D_τ onto D_γ where $\dim D_\tau = \dim D_\gamma$. Kaliman's theorem leads us to the following immediate corollary.

Theorem 2.6. *The group $Aut_\sharp(\overline{M}_n)$ is S_n.*

Proof. Let $\psi \in Aut_\sharp(\overline{M}_n)$. The restriction of ψ to the open stratum gives the permutation action on M_n since the automorphism group of the open stratum M_n contains only permutations ψ_ϱ. The unicity theorem of holomorphic maps implies that $\psi = \psi_\varrho$ since they coincide on the open stratum $\psi|_{M_n} = \psi_\varrho|_{M_n}$. □

Remark 2.1. Note that the whole group of holomorphic automorphisms $Aut(\overline{M}_n)$ is not necessarily isomorphic to S_n. For example, the automorphism group of $\overline{M}_4$ is $PSL_2(\mathbb{C})$. To the best of our knowledge, there is no systematic exposition of $Aut(\overline{M}_n)$ for $n > 5$.

3. Moduli of pointed real curves of genus zero

A *real structure* on complex variety X is an anti-holomorphic involution $c_X : X \to X$. The fixed point set $\mathbb{R}X = \mathrm{Fix}(c_X)$ of the involution is called the *real part* of the variety (or of the real structure).

In this section, we introduce the moduli spaces of pointed real curves of genus zero as the fixed point sets of real structures on $\overline{M}_n$.

3.1. Real structures on $\overline{M}_n$

The moduli space $\overline{M}_n$ comes equipped with a natural real structure. The involution $c : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$ acts on $\overline{M}_n$. Here a complex curve Σ is regarded as a pair $\Sigma = (C, J)$, where C is the underlying 2-dimensional manifold and J is a complex structure on it, and $\overline{\Sigma} = (C, -J)$ is its complex conjugated pair.[1]

Lemma 3.1. *The map c is a real structure on $\overline{M}_n$.*

Proof. The differentiability of c follows form the Kodaira-Spencer construction of infinitesimal deformations. We need to show that the differential of c is anti-linear at each $(\Sigma; \mathbf{p}) \in \overline{M}_n$. It is sufficient to show that it is anti-linear without taking the quotient with respect to $PSL_2(\mathbb{C})$.

The infinitesimal deformations of a nodal curve Σ with divisor $D_\mathbf{p} = p_1 + \cdots + p_n$ is canonically identified with the complex vector space $Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma)$, (see Section 2.2). By reversing the complex structure on Σ, we reverse the complex structure on the tangent space $Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma)$ at $(\Sigma; \mathbf{p})$. The differential of the map $(\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma) \to Ext^1_{\mathcal{O}_{\overline{\Sigma}}}(\Omega_{\overline{\Sigma}}(D_\mathbf{p}), \mathcal{O}_{\overline{\Sigma}})$$

is clearly anti-linear. □

[1] There is some notational ambiguity: The bar over $\overline{M}_n$ and that over $\overline{\Sigma}$ refer to different structures on underlying manifolds: the first one refers to the compactification of $\overline{M}_n$ and second refers to the manifold with reverse complex structure. Both of these notations are widely used, we use the bar for both cases. The context should make it clear which structure is referred to.

The subgroup $Aut_\sharp(\overline{M}_n) \cong S_n$ of holomorphic automorphisms acts on $\overline{M}_n$ via relabeling as given in (3). For each involution $\sigma \in S_n$, we have an additional real structure on $\overline{M}_n$:

$$c_\sigma := c \circ \psi_\sigma : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \sigma(\mathbf{p})). \tag{4}$$

Lemma 3.2. *Every real structure of $\overline{M}_n$ preserving the stratification is of the form (4) where $\sigma \in S_n$ is an involution.*

Proof. By their definition, anti-holomorphic automorphisms of $\overline{M}_n$ are obtained by composing the principal real structure $c : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$ with elements of $Aut(\overline{M}_n)$. The real structure c maps each stratum of $\overline{M}_n$ onto itself. Therefore, an anti-holomorphic automorphism $c \circ \psi$ respects the stratification of $\overline{M}_n$ if and only if $\psi \in Aut(\overline{M}_n)$ respects the stratification of $\overline{M}_n$ i.e., each real structure preserving the stratification of $\overline{M}_n$ is given by a certain involution $\sigma \in S_n$ and is of the form (4). □

3.2. σ-invariant curves and σ-equivariant families

Definition 3.1. An n-pointed stable curve $(\Sigma; \mathbf{p})$ is called *σ-invariant* if it admits a real structure $conj : \Sigma \to \Sigma$ such that $conj(p_i) = p_{\sigma(i)}$ for all $i \in \{1, \cdots, n\}$.

A family of n-pointed stable curves $\pi_S : \mathcal{U}_S \to S$ is called *σ-equivariant* if there exist a pair of real structures

$$\begin{array}{ccc} \mathcal{U}_S & \xrightarrow{c_\mathcal{U}} & \mathcal{U}_S \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi_S} \\ S & \xrightarrow{c_S} & S. \end{array}$$

such that the fibers $\pi^{-1}(s)$ and $\pi^{-1}(c_S(s))$ are Σ and $\overline{\Sigma}$ respectively, and $c_\mathcal{U}$ maps $z \in \pi^{-1}(s)$ to $z \in \pi^{-1}(c_S(s))$.

Remark 3.1. If $(\Sigma; \mathbf{p})$ is σ-invariant, then the real structure $conj : \Sigma \to \Sigma$ is uniquely determined by the permutation σ due to Lemma 2.2.

Lemma 3.3. *If $\pi : \mathcal{U}_S \to S$ is a σ-equivariant family, then each $(\Sigma; \mathbf{p}) \in \mathbb{R}S$ is σ-invariant.*

Proof. If $c_S((\Sigma; \mathbf{p})) = (\Sigma; \mathbf{p})$ then there exists a unique bi-holomorphic equivalence $conj : \Sigma \to \overline{\Sigma}$ (in other words, anti-holomorphic $conj : \Sigma \to \Sigma$) such that $conj(p_i) = p_{\sigma(i)}$. The restriction of $conj$ on labeled points is an involution. By applying Lemma 2.2 to $conj^2$, we determine that $conj$ is an involution on Σ. □

3.3. The moduli space of pointed real curves $\mathbb{R}\overline{M}_{(2k,l)}$

Let $\mathrm{Fix}(\sigma)$ be the fixed point set of the action of σ on the labeling set $\{1, \cdots, n\}$, and let $\mathrm{Perm}(\sigma)$ be its complement. Let $|\mathrm{Fix}(\sigma)| = l$ and $|\mathrm{Perm}(\sigma)| = 2k$.

Let $\varrho \in S_n$, and ψ_ϱ be the corresponding automorphism of $\overline{M}_n$. The conjugation of real structure c_σ with ψ_ϱ provides a conjugate real structure $c_{\sigma'} = \psi_\varrho \circ c_\sigma \circ \psi_{\varrho^{-1}}$.

The conjugacy classes of real structures are determined by the cardinalities $|\mathrm{Fix}(\sigma)|$ and $|\mathrm{Perm}(\sigma)|$. Therefore, from now on, we only consider c_σ where

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & 2k & 2k+1 & \cdots & 2k+l \\ k+1 & \cdots & 2k & 1 & \cdots & k & 2k+1 & \cdots & 2k+l \end{pmatrix}, \tag{5}$$

and $n = 2k + l$.

Definition 3.2. For σ as above in (5), σ-invariant curves are called *$(2k, l)$-pointed real curves.*

The fixed point set $\mathrm{Fix}(c_\sigma)$ is called the *moduli space of $(2k, l)$-pointed real curves* and denoted by $\mathbb{R}\overline{M}_{(2k,l)}$.

Theorem 3.4. *(i) For any $n \geq 3$, $\mathbb{R}\overline{M}_{(2k,l)}$ is a smooth real projective manifold of dimension $n - 3$.*

(ii) The universal family of curves $\pi : \overline{\mathcal{U}}_n \to \overline{M}_n$ is a σ-equivariant family.

(iii) Any σ-equivariant family of n-pointed stable curves over $\pi_S : \mathcal{U}_S \to S$ is induced by a unique pair of real morphisms

$$\begin{array}{ccc} \mathcal{U}_S & \xrightarrow{\hat{\kappa}} & \overline{\mathcal{U}}_n \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\kappa} & \overline{M}_n. \end{array}$$

(iv) Let $\mathfrak{M}_\sigma$ be the contravariant functor that sends real varieties (B, c_B) to the set of σ-equivariant families of curves over B. The moduli functor $\mathfrak{M}_\sigma$ is represented by the real variety $(\overline{M}_n, c_\sigma)$.

(v) Let $\mathbb{R}\mathfrak{M}_\sigma$ be the contravariant functor that sends real analytic manifolds R to the set of families of σ-invariant curves over R. The moduli functor $\mathbb{R}\mathfrak{M}_\sigma$ is represented by the real part $\mathbb{R}\overline{M}_{(2k,l)}$ of $(\overline{M}_n, c_\sigma)$.

Proof. (i) The smoothness of the real part of c_σ is a consequence of the implicit function theorem, and $dim_{\mathbb{R}}\ \mathbb{R}\overline{M}_{(2k,l)} = dim_{\mathbb{C}}\ \overline{M}_n = n - 3$ since the real part $\mathbb{R}\overline{M}_{(2k,l)}$ is not empty.

(ii) The fiber over $(\Sigma; \mathbf{p}) \in \overline{M}_n$ is $\pi^{-1}((\Sigma; \mathbf{p})) = \Sigma$. We define real structures on $\overline{M}_n$ and $\overline{\mathcal{U}}_n$ as follows;

$$\begin{array}{rlcl} c_\sigma : & (\Sigma; \mathbf{p}) & \mapsto & (\overline{\Sigma}; \sigma(\mathbf{p})), \\ \hat{c}_\sigma : & z \in \pi^{-1}((\Sigma; \mathbf{p})) & \mapsto & z \in \pi^{-1}((\overline{\Sigma}; \sigma(\mathbf{p}))), \end{array}$$

The real structures $c_\sigma, \hat{c}_\sigma$ satisfy the conditions of σ-equivariant families in Definition 3.1.

(iii) Due to Knudsen's theorem (see Section 2.3), each of the morphisms $\kappa : S \to \overline{M}_n$ and $\hat{\kappa} : \mathcal{U}_S \to \overline{\mathcal{U}}_n$ are unique. Therefore, they are the same as $c_\sigma \circ \kappa \circ c_S : S \to \overline{M}_n$ and $\hat{c}_\sigma \circ \hat{\kappa} \circ c_{\mathcal{U}} : \mathcal{U}_S \to \overline{\mathcal{U}}_n$. Hence, the morphisms $\kappa, \hat{\kappa}$ are real.

(iv)-(v) The statements follows from (iii) and the definition of moduli functors. □

It was believed that the real locus $\mathbb{R}\overline{M}_{(2k,l)}$ does not represent any moduli functor for $k \neq 0$, but it has only a meaning in operadic setting. Theorem 3.4 shows the contrary.

4. Stratification of $\mathbb{R}\overline{M}_{(2k,l)}$

A stratification for $\mathbb{R}\overline{M}_{(2k,l)}$ can be obtained by the stratification of $\overline{M}_n$ given in Section 2.3.

Lemma 4.1. *Let γ and $\overline{\gamma}$ be the dual trees of $(\Sigma;\mathbf{p})$ and $(\overline{\Sigma};\sigma(\mathbf{p}))$ respectively.*

(i) If γ and $\overline{\gamma}$ are not isomorphic, then the restriction of c_σ on the union of complex strata $D_\gamma \bigcup D_{\overline{\gamma}}$ gives a real structure with empty real part.

(ii) If γ and $\overline{\gamma}$ are isomorphic, then the restriction of c_σ on D_γ gives a real structure whose corresponding real part $\mathbb{R}D_\gamma$ is the intersection of $\mathbb{R}\overline{M}_{(2k,l)}$ with D_γ.

Proof. (i) Since γ and $\overline{\gamma}$ are not isomorphic, D_γ and $D_{\overline{\gamma}}$ are disjoint complex strata. The restriction of c_σ on $D_\gamma \bigcup D_{\overline{\gamma}}$ swaps the strata. Therefore, the real part of this real structure is empty.

(ii) Since γ and $\overline{\gamma}$ are isomorphic, the n-pointed curves $(\Sigma;\mathbf{p})$ and $(\overline{\Sigma};\sigma(\mathbf{p}))$ are in the same stratum D_γ. Therefore, the restriction of c_σ on D_γ is a real structure. The real part $\mathbb{R}D_\gamma$ of the σ-equivariant family D_γ is $\mathbb{R}\overline{M}_{(2k,l)} \bigcap D_\gamma$ since $\mathbb{R}\overline{M}_{(2k,l)} = \mathrm{Fix}(c_\sigma)$. □

Definition 4.1. A tree γ is called *σ-invariant* if it is isomorphic to $\bar{\gamma}$. We denote the set of σ-invariant n-trees by $\mathcal{T}ree(\sigma)$.

Theorem 4.2. $\mathbb{R}\overline{M}_{(2k,l)}$ *is stratified by real analytic subsets $\mathbb{R}D_\gamma$ where $\gamma \in \mathcal{T}ree(\sigma)$.*

Although the notion of σ-invariant trees leads us to a combinatorial stratification of $\mathbb{R}\overline{M}_{(2k,l)}$ as given in Theorem 4.2, it does not give a stratification in terms of connected strata. For a σ-invariant γ, the real part of the stratum $\mathbb{R}D_\gamma$ has many connected components. In the next subsection, we refine this stratification by using the spaces of $\mathbb{Z}_2$-equivariant point configurations in the projective line $\mathbb{CP}^1$.

4.1. Spaces of $\mathbb{Z}_2$-equivariant point configurations in $\mathbb{CP}^1$

Let $z := [z : 1]$ be the affine coordinate on $\mathbb{CP}^1$. Consider the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{CP}^1 \mid \Im(z) > 0\}$ (resp. lower half plane $\mathbb{H}^- = \{z \in \mathbb{CP}^1 \mid \Im(z) < 0\}$) as a half of the $\mathbb{CP}^1$ with respect to $z \mapsto \bar{z}$, and the real part $\mathbb{RP}^1$ as its boundary. Denote by $\mathbb{H}$ the compact disc $\mathbb{H}^+ \cup \mathbb{RP}^1$.

4.1.1. Irreducible $(2k,l)$-pointed real curves and configuration spaces

Let $(\Sigma;\mathbf{p})$ be an irreducible $(2k,l)$-pointed real curve. As a real curve, Σ is isomorphic to $\mathbb{CP}^1$ with real structure which is either $z \mapsto \bar{z}$ or $z \mapsto -1/\bar{z}$ (see, for example [19]). The real structure $z \mapsto -1/\bar{z}$ has empty real part. Notice that a $(2k,l)$-pointed curve with empty real part is possible only when $\mathrm{Fix}(\sigma) = \emptyset$ i.e., $l = 0$.

We will consider the spaces of real curves with non-empty and empty real parts as separate cases.

Case I. ***Configurations in*** $\mathbb{CP}^1$ ***with non-empty real part.*** Each finite subset $\mathbf{p}$ of $\mathbb{CP}^1$ which is invariant under the real structure $z \mapsto \bar{z}$ inherits additional structures:

I. *An oriented cyclic ordering on* $\mathrm{Fix}(\sigma)$: For any point $p \in (\mathbf{p} \bigcap \mathbb{RP}^1)$ there is unique $q \in (\mathbf{p} \bigcap \mathbb{RP}^1)$ which follows the point p in the positive direction of $\mathbb{RP}^1$ (the direction in which the coordinate $x := [x : 1]$ on $\mathbb{RP}^1$ increases).
The elements of $\mathbf{p}$ are labeled, therefore the cyclic ordering can be seen as a linear ordering on $(\mathbf{p} \bigcap \mathbb{RP}^1) \backslash \{p_n\}$. This linear ordering gives an *oriented cyclic ordering* on $\mathrm{Fix}(\sigma) = \{2k+1, \cdots, n\}$ which we denote by $\{i_1\} < \cdots < \{i_{l-1}\} < \{i_l\}$ where $i_l = n$.

II. *A 2-partition on* $\mathrm{Perm}(\sigma)$: The subset $\mathbf{p} \bigcap (\mathbb{CP}^1 \setminus \mathbb{RP}^1)$ of $\mathbf{p}$ admits a partition into two disjoint subsets $\{p_i \in \mathbb{H}^{\pm}\}$. This partition gives an *ordered 2-partition* $\mathrm{Perm}^{\pm} := \{i \mid p_i \in \mathbb{H}^{\pm}\}$ of $\mathrm{Perm}(\sigma)$. The subsets $\mathrm{Perm}^{\pm}$ are swapped by the permutation σ.

The set of data

$$o := \{(\mathbb{CP}^1, z \mapsto \bar{z}); \mathrm{Perm}^{\pm}; \mathrm{Fix}(\sigma) = \{\{i_1\} < \cdots < \{i_l\}\}\}$$

is called the *oriented combinatorial type* of the $\mathbb{Z}_2$-equivariant point configuration $\mathbf{p}$ on $(\mathbb{CP}^1, z \mapsto \bar{z})$.

The oriented combinatorial types of equivariant point configurations on $(\mathbb{CP}^1, z \mapsto \bar{z})$ enumerate the connected components of the space $\widetilde{Conf}_{(2k,l)}$ of k distinct pairs of conjugate points on $\mathbb{H}^+ \bigcup \mathbb{H}^-$ and l distinct points on $\mathbb{RP}^1$:

$$\begin{aligned} \widetilde{Conf}_{(2k,l)} \quad := \quad & \{(p_1, \cdots, p_{2k}; q_{2k+1}, \cdots, q_{2k+l}) \mid p_i \in \mathbb{CP}^1 \setminus \mathbb{RP}^1,\ p_i = p_j \Leftrightarrow i = j, \\ & p_i = \bar{p}_j \Leftrightarrow i = \sigma(j)\ \&\ q_i \in \mathbb{RP}^1,\ q_i = q_j \Leftrightarrow i = j\}. \end{aligned}$$

The number of connected components of $\widetilde{Conf}_{(2k,l)}$ is $2^k(l-1)!$.[2] They are all pairwise diffeomorphic; natural diffeomorphisms are given by σ-invariant relabeling.

Let $z := [z : 1]$ be the affine coordinate on $\mathbb{CP}^1$ and $x := [x : 1]$ be affine coordinate on $\mathbb{RP}^1$. The action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by

$$SL_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \quad (\Lambda, z) \mapsto \Lambda(z) = \frac{az+b}{cz+d}, \quad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

in affine coordinates. It induces an isomorphism $SL_2(\mathbb{R})/\pm I \to Aut(\mathbb{H})$. The automorphism group $Aut(\mathbb{H})$ acts on $\widetilde{Conf}_{(2k,l)}$

$$\Lambda : (z_1, \cdots, z_{2k}; x_{2k+1}, \cdots, x_{2k+l}) \mapsto (\Lambda(z_1), \cdots, \Lambda(z_{2k}); \Lambda(x_{2k+1}), \cdots, \Lambda(x_{2k+l})).$$

It preserves each of the connected components of $\widetilde{Conf}_{(2k,l)}$. This action is free when $2k + l \geq 3$, and it commutes with diffeomorphisms given by σ-invariant relabellings. Therefore, the quotient space $\widetilde{C}_{(2k,l)} := \widetilde{Conf}_{(2k,l)}/Aut(\mathbb{H})$ is a manifold of dimension $2k + l - 3$ whose connected components are pairwise diffeomorphic.

[2] Here we use the convention $n! = 1$ whenever $n \leq 0$.

In addition to the automorphisms considered above, there is a diffeomorphism $-\mathbb{I}$ of $\widetilde{Conf}_{(2k,l)}$ which is given in affine coordinates as follows.

$$-\mathbb{I} : (z_1, \cdots, z_{2k}; x_{2k+1}, \cdots, x_{2k+l}) \mapsto (-z_1, \cdots, -z_{2k}; -x_{2k+1}, \cdots, -x_{2k+l}). \tag{6}$$

Consider the quotient space $Conf_{(2k,l)} = \widetilde{Conf}_{(2k,l)}/(-\mathbb{I})$. Note that, $-\mathbb{I}$ interchanges components with *reverse* combinatorial types. Namely, the combinatorial type $\overline{o}$ of $-\mathbb{I}(\mathbf{p})$ is obtained from the combinatorial type o of $\mathbf{p}$ by reversing the cyclic ordering on $\mathrm{Fix}(\sigma)$ and swapping Perm^+ and Perm^-. The equivalence classes of oriented combinatorial types with respect to the action of $-\mathbb{I}$ are called *un-oriented combinatorial types* of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto \overline{z})$. The un-oriented combinatorial types enumerate the connected components of $Conf_{(2k,l)}$.

The diffeomorphism $-\mathbb{I}$ commutes with each σ-invariant relabeling and normalizing action of $Aut(\mathbb{H})$. Therefore, the quotient space $C_{(2k,l)} := Conf_{(2k,l)}/Aut(\mathbb{H})$ is a manifold of dimension $2k + l - 3$, its connected components are diffeomorphic to the components of $\widetilde{C}_{(2k,l)}$, and, moreover, the quotient map $\widetilde{C}_{(2k,l)} \to C_{(2k,l)}$ is a trivial double covering.

Case II. ***Configurations in*** $\mathbb{CP}^1$ ***with empty real part.*** Let $(\Sigma; \mathbf{p})$ be an irreducible $(2k, 0)$-pointed real curve and let $\mathbb{R}\Sigma = \emptyset$. Such a pointed real curve is isomorphic to $(\mathbb{CP}^1, \mathbf{p})$ with real structure $conj : z \mapsto -1/\bar{z}$.

The group of automorphisms of $\mathbb{CP}^1$ which commutes with $conj$ is

$$Aut(\mathbb{CP}^1, conj) \cong SU(2) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}.$$

Thus, the group $Aut(\mathbb{CP}^1, conj)$ acts naturally on the space

$$Conf^{\emptyset}_{(2k,0)} := \{(z_1, \cdots, z_{2k}) \mid z_i = -1/\bar{z}_{i+k}\}$$

of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$. For $k \geq 2$, the action is free and the quotient $B_{(2k,0)} := Conf^{\emptyset}_{(2k,0)}/Aut(\mathbb{CP}^1, conj)$ is a $2k - 3$ dimensional connected manifold.

The *combinatorial type* of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$ is unique and given by the topological type of the real structure $z \mapsto -1/\bar{z}$.

4.1.2. A normal position of $\mathbb{Z}_2$-equivariant point configurations on $\mathbb{CP}^1$

By using the automorphisms we can make the following choices for the representatives of the points in $\widetilde{C}_{(2k,l)}$ and $B_{(2k,0)}$.

Case I. ***Configurations in*** $\mathbb{CP}^1$ ***with non-empty real part.*** Every element in $\widetilde{C}_{(2k,l)}$ is represented by $(\mathbb{CP}^1, \mathbf{p})$ with $\mathbf{p} \in \widetilde{Conf}_{(2k,l)}$. In order to calibrate the choice by $Aut(\mathbb{H})$, consider an isomorphism $(\mathbb{CP}^1, \mathbf{p}) \mapsto (\mathbb{CP}^1, \mathbf{p}')$ which puts the labeled points in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

(A) In the case $l \geq 3$, the three consecutive labeled points $(p'_{i_{l-1}}, p'_n, p'_{i_1})$ in $\mathbb{RP}^1$ are put in the position $x'_{i_{l-1}} = 1, x'_n = \infty, x'_{i_1} = 0$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_k, \overline{z}_1, \cdots, \overline{z}_k, x_{2k+1}, \cdots, x_{2k+l-1}, \infty).$$

(B) In the case $l = 1, 2$, the three labeled points $\{p_k, p_{2k}, p_n\}$ are put in the position $\{\pm\sqrt{-1},\ \infty\}$. Then,

$$\mathbf{p}' = \begin{cases} (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-1}, -\epsilon\sqrt{-1}, x_{2k+1}, \infty) & \text{if } l = 2, \\ (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-1}, -\epsilon\sqrt{-1}, \infty) & \text{if } l = 1 \end{cases}$$

where $\epsilon = \pm$.

(C) In the case $l = 0$, the labeled points $\{p_k, p_{2k}\}$ are fixed at $\{\pm\sqrt{-1}\}$ and p_i where $\{i\} = \{k-1, 2k-1\} \bigcap \mathrm{Perm}^+$ is placed on the interval $]0, \sqrt{-1}[\subset \mathbb{H}^+$. Then,

$$\mathbf{p}' = (z_1, \cdots, z_{k-2}, \epsilon_1\lambda\sqrt{-1}, \epsilon_2\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-2}, -\epsilon_1\lambda\sqrt{-1}, -\epsilon_2\sqrt{-1}).$$

where $\lambda \in]0, 1[$ and $\epsilon_i = \pm, i = 1, 2$.

Remark 4.1 $(\mathbf{A}')$**.** In case of $k > 1$ and $l > 3$, we can consider the alternative map which puts the labeled points $\mathbf{p}'$ into the following normal positions. In this case, three labeled points $\{p_k, p_{2k}, p_n\}$ can be put at $\{z_k, z_{2k}, x_n\} = \{\pm\sqrt{-1}, \infty\}$ by the action of $Aut(\mathbb{H})$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-1}, -\epsilon\sqrt{-1}, x_{2k+1}, \cdots, x_{2k+l-1}, \infty)$$

where $\epsilon = \pm$.

Case II. ***Configurations in*** $\mathbb{CP}^1$ ***with empty real part.*** Let $k \geq 2$. Every element of $B_{(2k,0)}$ is represented by $(\mathbb{CP}^1, \mathbf{p})$ with $\mathbf{p} \in Conf^{\emptyset}_{(2k,0)}$. Calibrating the choice by $Aut(\mathbb{CP}^1, conj)$, consider an isomorphism $(\mathbb{CP}^1, \mathbf{p}) \mapsto (\mathbb{CP}^1, \mathbf{p}')$ which puts the labeled points of $(\Sigma; \mathbf{p})$ in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

(D)

$$\mathbf{p} = (z_1, \cdots, z_{k-2}, \lambda\sqrt{-1}, \sqrt{-1}, \frac{-1}{\overline{z}_1}, \cdots, \frac{-1}{\overline{z}_{k-2}}, -\frac{\sqrt{-1}}{\lambda}, -\sqrt{-1})$$

where $\lambda \in]-1, 1[$.

4.2. O/U-planar trees: one-vertex case.

An *oriented planar (o-planar) structure* on the one-vertex n-tree τ is one of the two possible sets of data

$$o \ := \begin{cases} \{\mathbb{R}\Sigma \neq \emptyset; \text{a } \sigma-\text{equivariant two}-\text{partition } \mathrm{Perm}^{\pm} \text{ of } \mathrm{Perm}(\sigma); \\ \text{an oriented cyclic ordering on } \mathrm{Fix}(\sigma)\}, \\ \{\mathbb{R}\Sigma = \emptyset\}. \end{cases}$$

We denote the o-planar trees by (τ, o).

An *un-oriented planar (u-planar) structure* u on the one-vertex n-tree τ is a pair of reverse o-planar structures $\{o, \bar{o}\}$ when $\mathbb{R}\Sigma \neq \emptyset$, and equal to the o-planar structure when $o = \{\mathbb{R}\Sigma = \emptyset\}$. We denote the u-planar trees by (τ, u).

4.2.1. O/U-planar trees and connected components of configuration spaces

As shown in Section 4.1.1, each connected component of $C_{(2k,l)}$ for $l > 0$ (resp. $C_{(2k,0)} \bigcup B_{(2k,0)}$ for $l = 0$) is associated to a unique u-planar tree since the un-oriented combinatorial types of $\mathbb{Z}_2$-equivariant point configurations are encoded by the same set of data. We denote the connected components of $C_{(2k,l)}$ (and $C_{(2k,0)} \bigcup B_{(2k,0)}$) by $C_{(\tau,u)}$. Similarly, each connected component of $\widetilde{C}_{(2k,l)}$ is associated to a unique o-planar tree. We denote the connected components of $\widetilde{C}_{(2k,l)}$ by $C_{(\tau,o)}$.

4.2.2. Connected components of $\mathbb{R}M_{(2k,l)}$

Every $\mathbb{Z}_2$-equivariant point configuration defines a $(2k, l)$-pointed real curve. Hence, we define

$$\Xi : \bigsqcup_{(\tau,u)} C_{(\tau,u)} \to \mathbb{R}M_{(2k,l)} \tag{7}$$

which maps $\mathbb{Z}_2$-equivariant point configurations to the corresponding isomorphism classes of irreducible $(2k, l)$-pointed curves.

Lemma 4.3. *(i) The map Ξ is a diffeomorphism.*
(ii) The configuration space $C_{(\tau,u)}$ is diffeomorphic to

- $((\mathbb{H}^+)^k \setminus \Delta) \times \Box^{l-3}$ *when* $l > 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}\})^{k-1} \setminus \Delta) \times \Box^{l-1}$ *when* $l = 1, 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}, \sqrt{-1}/2\})^{k-2} \setminus \Delta) \times \Box^1$ *when* $l = 0$ *and the type of real structure is* $(\mathbb{CP}^1, z \mapsto \bar{z})$,
- $((\mathbb{CP}^1 \setminus \{\sqrt{-1}, \sqrt{-1}/2, -\sqrt{-1}/2, -\sqrt{-1}\})^{k-2} \setminus (\Delta \cup \Delta^c)) \times \Box^1$ *when* $l = 0$ *and the type of the real structure is* $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$.

Here, Δ is the union of all diagonals $z_i \neq z_j (i \neq j)$, Δ^c is the union of all cross-diagonals $z_i \neq -\frac{1}{\bar{z}_j} (i \neq j)$, and $\Box^l$ is the l-dimensional open simplex.

Proof. (i) The map Ξ is clearly smooth. It is surjective since any $(2k, l)$-pointed irreducible curve is isomorphic either to $(\mathbb{CP}^1, z \mapsto \bar{z})$ or $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$ with a $\mathbb{Z}_2$-equivariant point configuration $\mathbf{p}$ on it. It is injective since the group of holomorphic automorphisms commuting with the real structure $z \mapsto \bar{z}$ is generated by $Aut(\mathbb{H})$ and $-\mathbb{I}$, and the group of holomorphic automorphisms commuting with the real structure $z \mapsto -1/\bar{z}$ is $Aut(\mathbb{CP}^1, conj)$. These automorphisms are taken into account during construction of the configuration spaces.

(ii) As it is shown in Section 4.1.1, $C_{(\tau,u)}$ is the quotient $C_{(\tau,o)} \bigsqcup C_{(\tau,\bar{o})}/(-\mathbb{I})$. The spaces $C_{(\tau,u)}$ and $C_{(\tau,o)}$ are clearly diffeomorphic. To replace $C_{(\tau,u)}$ by $C_{(\tau,o)}$, we choose an o-planar representative for each one-vertex u-planar tree (τ, u) among $(\tau, o), (\tau, \bar{o})$ as follows.

- $l \geq 3$ case: Let (τ, o) be the representative of (τ, u) for which $p_{2k+1} < p_{n-1} < p_n$ with respect to the cyclic ordering on $\mathrm{Fix}(\sigma)$.
- $l = 0, 1, 2$ case: Let (τ, o) be the representative of (τ, u) for which $k \in \mathrm{Perm}^+$.

We put the $\mathbb{Z}_2$-equivariant point configurations into a normal position as in 4.1.2. The parameterizations stated in (ii) for $l > 0$ cases follow from (**A**) and (**B**) of 4.1.2. In the case of $l = 0$ and $(\mathbb{CP}^1, z \mapsto \bar{z})$, according to (**C**) the configuration space $C_{(\tau,o)}$ is a locally trivial fibration over $\square^1 =]0,1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{H}^+ \setminus \{\sqrt{-1}, \lambda\sqrt{-1}\})^{k-2} \setminus \Delta$. Similarly, in the case of $l = 0$ and $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$, according to (**D**) the configuration space $C_{(\tau,u)}$ is a locally trivial fibration over $\square^1 =]-1,1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{CP}^1 \setminus \{\sqrt{-1}, \lambda\sqrt{-1}, -\sqrt{-1}/\lambda, -\sqrt{-1}\})^{k-2} \setminus \Delta$. Since the bases of these locally trivial fibrations are contractible, they are trivial fibrations, and the result follows. $\square$

4.3. Pointed real curves and o-planar trees

This section extends the notions of o/u-planar structures to the all σ-invariant trees.

4.3.1. Notations

Let $(\Sigma; \mathbf{p}) \in \mathbb{R}D_\gamma$ for some $\gamma \in \mathcal{T}ree(\sigma)$ and $conj : \Sigma \to \Sigma$ be the real structure on Σ. We denote the set of real components $\{v \in V_\gamma \mid conj(\Sigma_v) = \Sigma_v\}$ of Σ by $V_\gamma^{\mathbb{R}}$. If this set is empty, then $\mathbb{R}\Sigma$ is an isolated real node; we call the edge of γ representing the isolated real node the *special invariant edge*.

Two vertices $v, \bar{v} \in V_\gamma \setminus V_\gamma^{\mathbb{R}}$ are said to be *conjugate* if the real structure $conj$ maps the components Σ_v and $\Sigma_{\bar{v}}$ onto each other. Similarly, we call the flags $f, \bar{f} \in F_\gamma \setminus \{f_* \in F_\gamma \mid p_{f_*} \in \mathbb{R}\Sigma\}$ conjugate if $conj$ swaps the corresponding special points $p_f, p_{\bar{f}}$.

4.3.2. O/U-planar trees: general case.

Let $\gamma \in \mathcal{T}ree(\sigma)$. Each $(2k, l)$-pointed real curve $(\Sigma; \mathbf{p}) \in \mathbb{R}D_\gamma$ with $\mathbb{R}\Sigma \neq \emptyset$ admits additional structures:

I. If $V_\gamma^{\mathbb{R}} \neq \emptyset$, one can fix an oriented combinatorial type of point configurations on each real component Σ_v. Namely, for $v \in V_\gamma^{\mathbb{R}}$ with $\mathbb{R}\Sigma \neq \emptyset$, an oriented combinatorial type o_v is given by an oriented cyclic ordering on the set of *invariant flags* $F_\gamma^{\mathbb{R}}(v) = \{f \mid p_f \in \mathbb{R}\Sigma_v\}$ and an ordered 2-partition $F_\gamma^{\pm}(v) = \{i \mid p_i \in \mathbb{H}^{\pm}\}$ of $F_\gamma(v) \setminus F_\gamma^{\mathbb{R}}(v)$. In the case of $V_\gamma^{\mathbb{R}} = \{v\}$ and $\mathbb{R}\Sigma = \emptyset$, the o-planar structure is simply given by the type of the real structure, i.e., $\mathbb{R}\Sigma_v = \emptyset$ (as in 4.1.1).

II. If $V_\gamma^{\mathbb{R}} = \emptyset$, then one can fix an ordering of the flags of the special invariant edge: $f_e \mapsto \pm$, $f^e \mapsto \mp$.

This additional structures motivate the following definition.

Definition 4.2. An *o-planar* structure on $\gamma \in \mathcal{T}ree(\sigma)$ is the following set of data,

$$o := \begin{cases} \{(\gamma_v, o_v) \mid v \in V_\gamma^{\mathbb{R}}\} & \text{when } V_\gamma^{\mathbb{R}} \neq \emptyset, \\ \{\text{ordering } \{f_e, f^e\} \to \{\pm\}\}, & \text{when } V_\gamma^{\mathbb{R}} = \emptyset, \end{cases}$$

where (γ_v, o_v) is an o-planar structure on the one-vertex tree γ_v for $v \in V_\gamma^{\mathbb{R}}$, and $e = (f_e, f^e)$ is the special invariant edge of γ when $V_\gamma^{\mathbb{R}} = \emptyset$.

Similarly, an *u-planar* structure on γ is the following set of data,

$$u := \begin{cases} \{(\gamma_v, u_v) \mid v \in V_\gamma^{\mathbb{R}}\} & \text{when } V_\gamma^{\mathbb{R}} \neq \emptyset, \\ \{\emptyset\} & \text{when } V_\tau^{\mathbb{R}} = \emptyset, \end{cases}$$

where (γ_v, u_v) is an u-planar structure on the one-vertex tree γ_v for $v \in V_\gamma^{\mathbb{R}}$. We denote u-planar trees by (γ, u).

4.3.3. Notations.

We associate the subsets of vertices $V_\gamma^{\pm}$ and flags $F_\gamma^{\pm}$ to every o-planar tree (γ, o) when $V_\gamma^{\mathbb{R}} \neq \emptyset$ as follows. Let $v_1 \in V_\gamma \setminus V_\gamma^{\mathbb{R}}$ and let $v_2 \in V_\gamma^{\mathbb{R}}$ be the closest invariant vertex to v_1 in $||\gamma||$. Let $f \in F_\gamma(v_2)$ be in the shortest path connecting the vertices v_1 and v_2. The set $V_\gamma^{\pm}$ is the subset of vertices $v_1 \in V_\gamma \setminus V_\gamma^{\mathbb{R}}$ such that the flag f (defined as above) is in $F_\gamma^{\pm}(v_2)$. The subset of flags $F_\gamma^{\pm}$ is defined as $\partial_\gamma^{-1}(V_\gamma^{\pm})$.

Similarly, we associate the subsets of vertices $V_\gamma^{\pm}$ and flags $F_\gamma^{\pm}$ to every o-planar tree (γ, o) when $V_\gamma^{\mathbb{R}} = \emptyset$: Let $f_e \mapsto +$, $\partial_\gamma(f_e) = v_e$ and $\partial_\gamma(f^e) = v^e$. The set V_γ^+ is the subset of vertices v that are closer to v_e than v^e in $||\tau||$. The complement $V_\gamma \setminus V_\gamma^+$ is denoted by V_γ^-. The subset of flags $F_\gamma^{\pm}$ is defined as $\partial_\gamma^{-1}(V_\gamma^{\pm})$.

4.4. O-planar trees and their configuration spaces

We associate a product of configuration spaces of $\mathbb{Z}_2$-equivariant point configurations $C_{(\tau_v, o_v)}$ and moduli space of pointed complex curves $\overline{M}_{|v|}$ to each o-planar tree (τ, o):

$$C_{(\tau,o)} \quad := \quad \begin{cases} \prod_{v \in V_\tau^{\mathbb{R}}} C_{(\tau_v, o_v)} \times \prod_{v \in V_\tau^+} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset \text{ and } \mathbb{R}\Sigma \neq \emptyset, \\ C_{(\tau_{v_r}, o_{v_r})} \times \prod_{\{v, \bar v\} \subset V_\tau \setminus V_\tau^{\mathbb{R}}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset \text{ and } \mathbb{R}\Sigma = \emptyset, \\ \prod_{v \in V_\tau^+} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} = \emptyset. \end{cases}$$

For the case $\mathbb{R}\Sigma = \emptyset$, v_r is the vertex corresponding to the unique real component of point curves, and the product runs over the un-ordered pairs of conjugate vertices belonging to $V_\tau \setminus V_\tau^{\mathbb{R}}$ i.e., $\{v, \bar v\} = \{\bar v, v\}$.

For each u-planar tree (τ, u), we first choose an o-planar representative and then put $C_{(\tau,u)} = C_{(\tau,o)}$. Note that the so defined space $C_{(\tau,u)}$ does not depend on the o-planar representatives up to an isomorphism.

Lemma 4.4. *Let $\gamma \in \mathcal{T}ree(\sigma)$. The real part $\mathbb{R}D_\gamma$ is diffeomorphic to $\bigsqcup_{(\gamma,u)} C_{(\gamma,u)}$ where the disjoint union is taken over all possible u-planar structures of γ.*

Proof. The complex stratum D_γ is diffeomorphic to the product $\prod_{v \in V_\gamma} \overline{M}_{|v|}$. The real structure $c_\sigma : D_\gamma \to D_\gamma$ maps the factor $\overline{M}_{|v|}$ onto $\overline{M}_{|\bar v|}$ when v and $\bar v$ are conjugate vertices, and maps the factor $\overline{M}_{|v|}$ onto itself when $v \in V_\gamma^{\mathbb{R}}$. Therefore, the real part $\mathbb{R}D_\gamma$

of c_σ is given by

$$\prod_{v\in V_\gamma^{\mathbb{R}}} C_{(2k_v,l_v)} \times \prod_{\{v,\bar v\}\subset V_\gamma\setminus V_\gamma^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when } |V_\gamma^{\mathbb{R}}|>1,$$

$$(C_{(|v_r|,0)} \bigsqcup B_{(|v_r|,0)}) \times \prod_{\{v,\bar v\}\subset V_\gamma\setminus V_\gamma^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when} |V_\gamma^{\mathbb{R}}|=1,$$

$$\prod_{\{v,\bar v\}\subset V_\gamma\setminus V_\gamma^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when} |V_\gamma^{\mathbb{R}}|=0,$$

where $k_v = |F_\gamma^+(v)|$ and $l_v = |F_\gamma^{\mathbb{R}}(v)|$. The decompositions of the spaces $C_{(2k,l)}$ and $C_{(2k,0)} \bigsqcup B_{(2k,0)}$ into their connected components are given in Lemma 4.3. □

Theorem 4.5. $\mathbb{R}\overline{M}_{(2k,l)}$ *is stratified by* $C_{(\gamma,u)}$.

Proof. The moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ can be stratified by $\mathbb{R}D_\gamma$ due to Theorem 4.2. The claim directly follows from the decompositions of open strata of $\mathbb{R}\overline{M}_{(2k,l)}$ into their connected components given in Lemma 4.4. □

4.5. Boundaries of the strata

In this section we investigate the adjacency of the strata $C_{(\gamma,u)}$ in $\mathbb{R}\overline{M}_{(2k,l)}$. We start with considering the complex situation in order to introduce natural coordinates near each codimension one stratum $C_{(\gamma,u)}$.

4.5.1. Intermezzo: Coordinates around the codimension one strata

Let γ be a 2-vertex n-tree given by $V_\gamma = \{v_e, v^e\}$, $F_\gamma(v^e) = \{i_1, \cdots, i_s, f^e\}$ and $F_\gamma(v_e) = \{f_e, i_{s+1}, \cdots, i_{n-1}, n\}$. Let $(z,w) := [z:1]\times[w:1]$ be affine coordinates on $\mathbb{CP}^1\times\mathbb{CP}^1$. Here we introduce coordinates around D_γ.

Consider a neighborhood $V \subset D_\gamma$ of a nodal n-pointed curve $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. Any $(\Sigma;\mathbf{p}) \in V$ can be identified with a nodal curve $\{(z-z_{f_e})\cdot(w-w_{f^e})=0\}$ in $\mathbb{CP}^1\times\mathbb{CP}^1$ with special points $\mathbf{p}_{v_e} = (a_{f_e}, a_{i_{s+1}}, \cdots, a_{i_{n-1}}, a_n) \subset \{w-w_{f^e}=0\}$ and $\mathbf{p}_{v^e} = (b_{f^e}, b_{i_1}, \cdots, b_{i_s}) \subset \{z-z_{f_e}=0\}$. In order to determine a nodal curve in $\mathbb{CP}^1\times\mathbb{CP}^1$ and the position of its special points uniquely defined by $(\Sigma;\mathbf{p})$, we make the following choice. Firstly, we fix three labeled points $a_{i_{s+1}}, a_{i_{n-1}}, a_n$ on the line $\{w-w_{f^e}=0\}$ whenever $|v_e|>3$, and three special points $a_{f_e}, a_{i_{n-1}}, a_n$ whenever $|v_e|=3$. Secondly, we fix three special points $b_{f^e}, b_{i_1}, b_{i_s}$ on $\{z-z_{f_e}=0\}$. Finally, we choose $a_{i_{s+1}}=(0,0)$, $a_{i_{n-1}}=(1,0)$, $a_n=(\infty,0)$ for $|v_e|>3$; $a_{f_e}=(0,0)$, $a_{i_{n-1}}=(1,0)$, $a_n=(\infty,0)$ for $|v_e|=3$; and $b_{i_1}=(z_{f_e},1)$, $b_{i_s}=(z_{f_e},\infty)$, $b_{f^e}=(z_{f_e},0)$. Then the components z and w of the special points provide a coordinate system in V; in particular, for $|v_e|>3$ such a coordinate system is formed by z_{f^e}, z_{i_*} with $i_* = i_{s+2}, \cdots, i_{n-2}$, and w_{j_*} with $j_* = i_2, \cdots, i_{s-1}$.

We now consider a family of n-pointed curves over V times the ϵ-ball $B_\epsilon = \{|t|<\epsilon\}$. It is given by a family curves $\{(z-z_{f_e})\cdot w + t = 0 \mid t \in B_\epsilon\}$ in $\mathbb{CP}^1\times\mathbb{CP}^1$. The labeled points

$(z_i, w_i), i = 1, \cdots n$ on these curves are chosen in the following way. If $|v_e| > 3$, we put $(z_{i_1}, w_{i_1}) = (z_{f_e} - t, 1)$, $(z_{i_s}, w_{i_s}) = (z_{f_e}, \infty)$, $(z_{i_{s+1}, w_{i_{s+1}}}) = (0, t/z_{f_e})$, $(z_{i_{n-1}}, w_{i_{n-1}}) = (1, -t/(1 - z_{f_e}))$ and $(z_n, w_n) = (\infty, 0)$. Similarly, for $|v_e| = 3$, $(z_{i_1}, w_{i_1}) = (-t, 1)$, $(z_{i_{n-2}}, w_{i_{n-2}}) = (0, \infty)$, $(z_{i_{n-1}}, w_{i_{n-1}}) = (1, -t)$ and $(z_n, w_n) = (\infty, 0)$. The other labeled points are taken in an arbitrary position. The component z of the special points and the parameter t provide a coordinate system in $V \times B_\epsilon$.

Due to Knudsen's theorem there exists a unique $\kappa : V \times B_\epsilon \to \overline{M}_n$ which gives the family of n-pointed curves given above.

Lemma 4.6. $\det(d\kappa) \neq 0$ *at* $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. *Hence,* κ *gives a local isomorphism.*

Proof. The parameter t gives a regular function on $\kappa(V \times B_\epsilon)$ which is vanishing along $D_\gamma \bigcap \kappa(V \times B_\epsilon)$. The differential $d\kappa(\vec{v}) = \vec{v}$ for $\vec{v} \in T_{(\Sigma^o, \mathbf{p}^o)} V$ since the restriction of κ on $V \times \{0\}$ is the identity map. We need to prove that $d\kappa(\partial_t) \neq 0$. In other words, the curves are non-isomorphic for different values of the parameter t. Let $(\Sigma(t_i), \mathbf{p}(t_i)) \in V \times B_\epsilon$ be two n-pointed curves for $t_1 \neq t_2$. A bi-holomorphic map $\Phi : \Sigma(t_1) \to \Sigma(t_2)$ is determined by the images of $p_{s+1}, p_{i_{n-1}}, p_n$ when $|v_e| > 3$, and by the images of $p_{i_{n-2}}, p_{i_{n-1}}, p_n$ when $|v_e| = 3$. However, the bi-holomorphic map Φ mapping $(p_{s+1}, p_{i_{n-1}}, p_n)(t_1) \mapsto (p_{s+1}, p_{i_{n-1}}, p_n)(t_2)$ (resp. $(p_{i_{n-2}}, p_{i_{n-1}}, p_n)(t_1) \mapsto (p_{i_{n-2}}, p_{i_{n-1}}, p_n)(t_2)$) maps $p_{i_1}(t_1) = (z_{f_e} - t_1, 1)$ to $(z_{f_e} - t_1, t_2/t_1) \neq p_{i_1}(t_2)$ (resp. $p_{i_1}(t_1) = (-t_1, 1)$ to $(-t_1, t_2/t_1) \neq p_{i_1}(t_2)$), i.e., Φ can not be an isomorphism. □

Remark 4.2. Due to Lemma 4.6, the coordinates on $V \times B_\epsilon$ provide a coordinate system at $(\Sigma^o; \mathbf{p}^o) \in D_\gamma$. There is a natural coordinate projection $\rho : V \times B_\epsilon \to V$.

For a σ-invariant γ and c_σ-invariant V, the above coordinates and the local isomorphism κ are equivariant with respect to a suitable real structure ($(z, w) \mapsto (\bar{z}, \bar{w})$ when $\mathbb{R}\Sigma \neq \emptyset$, and $(z, w) \mapsto (\bar{w}, \bar{z})$ when $\mathbb{R}\Sigma = \emptyset$) on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Therefore, the real part $\mathbb{R}V \times] - \epsilon, \epsilon[$ of $V \times B_\epsilon$ provides a neighborhood for a $(\Sigma^o; \mathbf{p}^o)$ in $\mathbb{R}D_\gamma$ with a set of coordinates on it.

4.5.2. Contraction morphisms for o-planar trees.

Let $(\gamma, \hat{o})$ be an o-planar tree and $\phi : \gamma \to \tau$ be a morphism of n-trees contracting an invariant set of edges $E_{con} = E_\gamma \setminus \phi_E(E_\tau)$. In such a situation, we associate a particular o-planar structure o on τ, as described below in separate cases **(a)** and **(b)**, and speak of a *contraction morphism* $\varphi : (\gamma, \hat{o}) \to (\tau, o)$. In all the cases, except **(a-2)**, the o-planar structure o is uniquely defined by $\hat{o}$.

(a) Let $E_{con} = \{e = (f_e, f^e)\}$ and e be an invariant edge.

(1) If $\partial_\gamma(e) = \{v_e, v^e\} \subset V_\gamma^{\mathbb{R}}$, then we convert the o-planar structures

$$\begin{aligned} \hat{o}_{v_e} &= \{\mathbb{R}\Sigma_{v_e} \neq \emptyset; F_\gamma^{\pm}(v_e); F_\gamma^{\mathbb{R}}(v_e) = \{\{i_1\} < \cdots < \{i_m\} < \{f_e\}\}\} \\ \hat{o}_{v^e} &= \{\mathbb{R}\Sigma_{v^e} \neq \emptyset; F_\gamma^{\pm}(v^e); F_\gamma^{\mathbb{R}}(v^e) = \{\{i'_1\} < \cdots < \{i'_{m'}\} < \{f^e\}\}\}. \end{aligned}$$

at v_e and v^e to an o-planar structure at vertex $v = \phi_V(\{v_e, v^e\})$ of (τ, o) defining it by

$$\begin{aligned} o_v &= \{\mathbb{R}\Sigma_v \neq \emptyset; F^{\pm}_{\tau}(v) = F^{\pm}_{\gamma}(v_e) \bigcup F^{\pm}_{\gamma}(v^e); \\ &\quad F^{\mathbb{R}}_{\tau}(v) = \{\{i_1\} < \cdots < \{i_m\} < \{i'_1\} < \cdots < \{i'_{m'}\}\}\}. \end{aligned}$$

The o-planar structures are kept unchanged at all other invariant vertices.

(2) If e is a special invariant edge, then we convert the o-planar structure $\hat{o} = \{f_e \mapsto +, f^e \mapsto -\}$ of γ into an o-planar structure at the vertex $v = \phi_V(\{v_e, v^e\})$ of τ defining it by

$$o_v = \{\mathbb{R}\Sigma_v \neq \emptyset; F^{+}_{\tau}(v) = F^{+}_{\gamma}(v_e) \setminus \{f_e\}, F^{-}_{\tau}(v) = F^{-}_{\gamma}(v^e) \setminus \{f^e\}; F^{\mathbb{R}}_{\tau}(v) = \emptyset\}$$

or by

$$o_v = \{\mathbb{R}\Sigma_v = \emptyset\}.$$

(b) Let $E_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1, 2\}$ where $f_{e_i}, i = 1, 2$ and $f^{e_i}, i = 1, 2$ are conjugate pairs of flags.

(1) If $\partial_\gamma(e_i) = \{\hat{v}, v^{e_i}\}$, and $\hat{v} \in V^{\mathbb{R}}_{\gamma}$, $v^{e_i} \notin V^{\mathbb{R}}_{\gamma}$, then we convert the o-planar structure

$$o_{\hat{v}} = \{\mathbb{R}\Sigma_{\hat{v}} \neq \emptyset; F^{\pm}_{\gamma}(\hat{v}); F^{\mathbb{R}}_{\gamma}(\hat{v}) = \{\{i_1\} < \cdots < \{i_m\}\}\}.$$

at $\hat{v}$ to an o-planar structure at $v = \phi(\{\hat{v}, v^{e_1}, v^{e_2}\})$ of τ defining it by

$$\begin{aligned} o_v &= \{\mathbb{R}\Sigma_v \neq \emptyset; F^{+}_{\tau}(v) = F^{+}_{\gamma}(\hat{v}) \bigcup F^{+}_{\gamma}(v^{e_1}) \setminus \{f_{e_1}, f^{e_1}\}, \\ &\quad F^{-}_{\tau}(v) = F^{-}_{\gamma}(\hat{v}) \bigcup F^{-}_{\gamma}(v^{e_2}) \setminus \{f_{e_2}, f^{e_2}\}; \\ &\quad F^{\mathbb{R}}_{\tau}(v) = \{\{i_1\} < \cdots < \{i_m\}\}\}. \end{aligned}$$

(2) If $\partial_\gamma(e_i) = \{\hat{v}, v^{e_i}\}$, and $\hat{v} \in V^{\mathbb{R}}_{\gamma}$, $v^{e_i} \notin V^{\mathbb{R}}_{\gamma}$, then we convert the o-planar structure $o_{\hat{v}} = \{\mathbb{R}\Sigma_{\hat{v}} = \emptyset\}$ at the vertex $\hat{v}$ to an o-planar structure at $v = \phi(\{\hat{v}, v^{e_1}, v^{e_2}\})$ of τ defining it by $o_v = \{\mathbb{R}\Sigma_v = \emptyset\}$.

(3) If $E_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1, 2\}$ and $\partial_\gamma(e_i) \bigcap V^{\mathbb{R}}_{\gamma} = \emptyset$, then we define the o-planar structure at each v in τ to be the same as the o-planar structure at v of $(\gamma, \hat{o})$.

Remark 4.3. Let $(\gamma, \hat{o})$ be an o-planar tree with $\mathbb{R}\Sigma \neq \emptyset$, and let $\varphi_e : (\gamma, \hat{o}) \to (\tau, o)$ be the contraction of an edge $e \in E_\gamma$. If the o-planar tree (τ, o) and the u-planar tree $(\gamma, \hat{u})$ underlying $(\gamma, \hat{o})$ are given, then the o-planar structure $\hat{o}$ can be reconstructed. For this reason, when a stratum $C_{(\gamma,\hat{u})}$ contained in the boundary of $\overline{C}_{(\tau,u)}$ is given, we denote the corresponding o-planar structure $\hat{o}$ by $\delta(o)$.

Proposition 4.7. *A stratum $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,u)}$ if and only if the u-planar structures $u, \hat{u}$ can be lifted to o-planar structures $o, \hat{o}$ in such a way that (τ, o) is obtained by contracting an invariant set of edges of $(\gamma, \hat{o})$.*

Proof. We need to consider the statement only for the strata of codimension one and two. These cases correspond to the contraction morphisms from two/three-vertex o-planar (sub)trees to one-vertex o-planar (sub)trees given in **(a)** and **(b)**. For a stratum of higher codimension, the statement can be proved by applying the elementary contractions **(a)** and **(b)** inductively. Here, we consider only the case **(a-1)**. The proof for other cases is the same.

We first assume that (τ, o) is obtained by contracting the edge e of $(\gamma, \hat{o})$, where $(\gamma, \hat{o})$ is an o-planar two-vertex tree with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$. An element $(\Sigma; \mathbf{p}) \in C_{(\gamma,\hat{o})}$ can be represented by the nodal curve $\{(z - z_{f_e}) \cdot w = 0\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with special points $a_f = (z_f, 0)$ and $b_f = (z_{f_e}, w_f)$ such that

$$\begin{array}{ll} a_f \in \{w = 0 \ \& \ \Im(z) > 0\} & \text{for } f \in F_\gamma^+(v_e) \\ a_{\bar{f}} \in \{w = 0 \ \& \ \Im(z) < 0\} & \text{for } \bar{f} \in F_\gamma^-(v_e) \\ \{a_{i_1} < \cdots < a_{i_m}\} \subset \{w = 0 \ \& \ \Im(z) = 0\} & \text{for } i_* \in F_\gamma^{\mathbb{R}}(v_e) \end{array}$$

on the axis $w = 0$, and $b_* = (z_{f_e}, w_*)$

$$\begin{array}{ll} b_f \in \{z - z_{f_e} = 0 \ \& \ \Im(w) > 0\} & \text{for } f \in F_\gamma^+(v^e) \\ b_{\bar{f}} \in \{z - z_{f_e} = 0 \ \& \ \Im(w) < 0\} & \text{for } \bar{f} \in F_\gamma^-(v^e) \\ \{b_{i'_1} < \cdots < b_{i'_{m'}}\} \subset \{z - z_{f_e} = 0 \ \& \ \Im(w) = 0\} & \text{for } i'_* \in F_\gamma^{\mathbb{R}}(v^e) \end{array}$$

When we include the curve $\{(z - z_{f_e}) \cdot w = 0\}$ into the family $\{(z - z_{f_e}) \cdot w + t = 0\}$, the complex orientation defined on the irreducible components $w = 0$ and $z - z_{f_e} = 0$ by the halves $\Im(z) > 0$ and, respectively, $\Im(w) > 0$ extends continuously to a complex orientation of $\{(z - z_{f_e}) \cdot w + t = 0\}$ with $t \in [0, \epsilon[$ defined by, say, $\Im(z) > 0$. As a result, the curves $\{(z - z_{f_e}) \cdot w + t = 0\}$ with $t \in [0, \epsilon[$ acquire an o-planar structure given by

$$\begin{array}{ll} (z_f, w_f) \in \{(z - z_{f_e}) \cdot w + t = 0 \ \& \ \Im(z) > 0\} & \text{for } f \in F_\gamma^+(v_e) \bigcup F_\gamma^+(v^e) \\ (z_{\overline{f}}, w_{\overline{f}}) \in \{(z - z_{f_e}) \cdot w + t = 0 \ \& \ \Im(z) < 0\} & \text{for } \bar{f} \in F_\gamma^-(v_e) \bigcup \in F_\gamma^-(v^e) \\ (z_f, w_f) \in \{(z - z_{f_e}) \cdot w + t = 0 \ \& \ \Im(z) = 0\} & \text{for } f \in F_\gamma^{\mathbb{R}}(v_e) \bigcup F_\gamma^{\mathbb{R}}(v^e) \end{array}$$

where the points on the real part of the curves $\{(z - z_{f_e}) \cdot w + t = 0\}$ are cyclicly ordered by

$$z_{i_1} < \cdots < z_{i_m} < z_{i'_1} < \cdots < z_{i'_{m'}}.$$

This is exactly the o-planar structure (τ, o) defined in **(a-1)** of 4.5.2.

Now assume that $C_{(\gamma,\hat{u})}$, where $(\gamma, \hat{u})$ is an u-planar tree with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, is contained in the boundary of $\overline{C}_{(\tau,u)}$. There are four different o-planar representatives of $(\gamma, \hat{u})$, and any two non reverse to each other representatives $\hat{o}_1, \hat{o}_2$ provide by contraction two different o-planar structures $(\tau, o_i), i = 1, 2$. By the already proved part of the statement, $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,o_i)}$ for each $i = 1, 2$. It remains to notice that any codimension one stratum is adjacent to at most two main strata. □

4.5.3. Examples

(i) The first nontrivial example is $\overline{M}_4$. There are three real structures: c_{σ_1}, c_{σ_2}, c_{σ_3}, where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

These real structures then give $\mathbb{R}\overline{M}_{(2k,l)}$, where $(2k,l) = (0,4), (2,2)$, and $(4,0)$ respectively.

In the case $\sigma = id$, $\mathbb{R}M_{(0,4)}$ is the configuration space of four distinct points on $\mathbb{RP}^1$ up to the action of $PSL_2(\mathbb{R})$. The 4-pointed curves $(\Sigma; \mathbf{p}) \in \mathbb{R}M_{(0,4)}$ can be identified with $(0, x_2, 1, \infty)$ where $x_2 \in \mathbb{RP}^1 \setminus \{0,1,\infty\}$. Hence, $\mathbb{R}M_{(0,4)} = \mathbb{RP}^1 \setminus \{0,1,\infty\}$ and its compactification is $\mathbb{R}\overline{M}_{(0,4)} = \mathbb{RP}^1$. The three intervals of $\mathbb{R}M_{(0,4)}$ are the three configuration spaces $C_{(\tau,u_i)}$ and the three points are the configuration spaces $C_{(\gamma_i,u_i)}$. The u-planar trees (τ, u_i) and (γ_i, u_i) are given in Fig. 2.

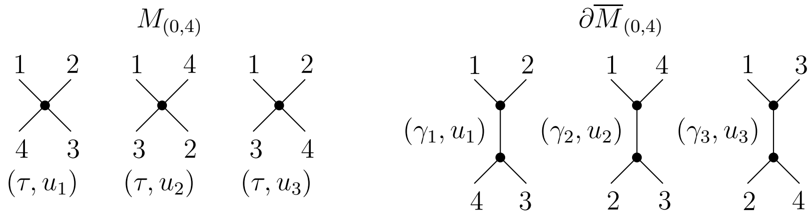

FIGURE 2. U-planar trees encoding the strata of $\overline{M}_{(0,4)}$.

In the case $\sigma = \sigma_1$, $\mathbb{R}M_{(2,2)}$ is the space of distinct configurations of two points in $\mathbb{RP}^1$ and a pair of complex conjugate points in $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. $(\Sigma; \mathbf{p}) \in \mathbb{R}M_{(2,2)}$ is identified with $(\sqrt{-1}, -\sqrt{-1}, x_3, \infty) \in C_{(\tau,u)}, -\infty < x_3 < \infty$. Hence, $\mathbb{R}M_{(2,2)} = \mathbb{RP}^1 \setminus \{\infty\}$ and its compactification is $\mathbb{R}\overline{M}_{(2,2)} = \mathbb{RP}^1$. The interval $\mathbb{R}M_{(2,2)}$ is $C_{(\tau,u)}$ and the point at its closure is $C_{(\gamma,u)}$.

In the case $\sigma = \sigma_2$, the space $\mathbb{R}M_{(4,0)}$ has different pieces parameterizing real curves with non-empty and empty real parts: The subspace of $\mathbb{R}M_{(4,0)}$ parameterizing the $(4,0)$-pointed real curve with $\mathbb{R}\Sigma \neq \emptyset$ is $(\lambda\sqrt{-1}, \sqrt{-1}, -\lambda\sqrt{-1}, -\sqrt{-1})$ where $\lambda \in]-1,1[\setminus\{0\}$. The subspace of $\mathbb{R}M_{(4,0)}$ parameterizing the real curves with $\mathbb{R}\Sigma = \emptyset$ is $(\lambda\sqrt{-1}, \sqrt{-1}, -\sqrt{-1}/\lambda, -\sqrt{-1})$, where $\lambda \in]-1,1[$. Note that, the pieces parameterizing $\mathbb{R}\Sigma \neq \emptyset$ and $\mathbb{R}\Sigma = \emptyset$ are joined through the boundary points corresponding to curves with isolated real singular points. The compactification $\mathbb{R}\overline{M}_{(4,0)}$ is $\mathbb{RP}^1$.

(ii) The moduli space $\overline{M}_5$ has three different real structures $c_{\sigma_1}, c_{\sigma_2}$ and c_{σ_3} where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}.$$

The space $\mathbb{R}M_{(0,5)}$ is identified with the configuration space of five distinct points on $\mathbb{RP}^1$ modulo $PSL_2(\mathbb{R})$. It is $(\mathbb{RP}^1 \setminus \{0,1,\infty\})^2 \setminus \Delta$, where Δ is union of all diagonals. Each connected component of $\mathbb{R}M_{(0,5)}$ is isomorphic to a two dimensional simplex. The closure of each cell can be obtained by adding the boundaries given in Section 4.5; for an example see Fig. 3a. It gives the compactification of $\mathbb{R}\overline{M}_{(0,5)}$ which is a torus with 3 points blown up: the cells corresponding to u-planar trees (τ, u_1) and (τ, u_2) are glued along the face corresponding to (γ, u) which gives $(\tau, u_i), i = 1, 2$ by contracting some edges, see Fig. 3b.

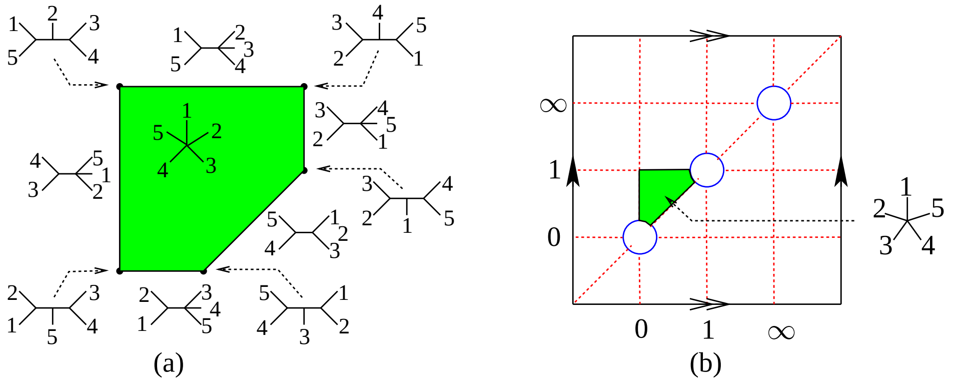

FIGURE 3. (a) Stratification of $\widetilde{C}_{(\tau,u)}$. (b) The stratification of $\mathbb{R}\overline{M}_{(0,5)}$.

The space $\mathbb{R}M_{(2,3)}$ is isomorphic to configurations of a conjugate pair of points on $\mathbb{CP}^1$. The automorphisms allows us to identify such configurations with $(z, \bar{z}, 0, 1, \infty)$ where $z \in \mathbb{C} \setminus \mathbb{R}$. Hence, it can be given as $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. The $\mathbb{R}\overline{M}_{(2,3)}$ is obtained as a sphere with 3 points blown up according to the stratification given in Section 4.5.

Finally, elements of $\mathbb{R}M_{(4,1)}$ can be identified with $(z, \sqrt{-1}, \bar{z}, -\sqrt{-1}, \infty)$. Hence it can be identified with $\mathbb{CP}^1 \setminus (\mathbb{RP}^1 \cup \{\sqrt{-1}, -\sqrt{-1}\})$. Therefore, connected components are isomorphic to $\mathbb{H}^+ \setminus \{\sqrt{-1}\}$. The $\mathbb{R}\overline{M}_{(4,1)}$ is a sphere with a point blown up.

The moduli space $\overline{M}_5$ is a del Pezzo surface of degree 5 and these are all the possible real parts of this del Pezzo surface (see [1]).

5. The first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$

In this section we calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ by using the stratification given in Theorem 4.5.

5.1. Orientations of top-dimensional strata

Let (τ, o) be a one-vertex o-planar tree. The coordinates on $C_{(\tau,o)}$ given in Section 4.1.2 determine an orientation of $C_{(\tau,o)}$. For instance, let $|\mathrm{Fix}(\sigma)| \geq 3$ and let the o-planar structure on (τ, o) be given by $\mathrm{Perm}^{\pm}$ and by a linear ordering $x_{i_1} = 0 < x_{i_2} < \cdots < x_{i_{l-1}} = 1 < x_{i_l} := x_n = \infty$ on $\mathrm{Fix}(\sigma)$. The coordinates in **(A)** of 4.1.2 generate the following top-dimensional differential form on $C_{(\tau,o)}$:

$$\omega_{(\tau,o)} := \left(\frac{\sqrt{-1}}{2}\right)^k \bigwedge_{\alpha_*\in\mathrm{Perm}^+} dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*} \bigwedge dx_{i_2}\wedge\cdots\wedge dx_{i_{l-2}}. \tag{8}$$

The multiplication of top-dimensional forms with a positive valued function $\Theta : C_{(\tau,o)} \to \mathbb{R}_{>0}$ defines an equivalence relation on sections of $\det(TC_{(\tau,o)})$. An *orientation* is an equivalence class of nowhere zero top-dimensional forms with respect to this equivalence relation. We denote the equivalence class of $\omega_{(\tau,o)}$ by $[\omega_{(\tau,o)}]$.

Similarly, using the coordinates given in **(A′)**, **(B)**, **(C)** and **(D)** in 4.1.2 and their ordering, we determine differential forms $\omega_{(\tau,o)}$ and orientations $[\omega_{(\tau,o)}]$ of $C_{(\tau,o)}$ for all (τ,o) with $|V_\tau| = 1$.

5.2. Orientations of codimension one strata

Let (γ,o) be a two-vertex o-planar tree. Let $V_\gamma = \{v_e, v^e\}$ and $e = (f_e, f^e)$ be the edge where $\partial_\gamma(n) = \partial_\gamma(f_e) = v_e$ and $\partial_\gamma(f^e) = v^e$.

By choosing three flags in $F_\gamma(v_e)$ and $F_\gamma(v^e)$, and using the calibrations as in 4.1.2 we obtain a coordinate system in $C_{(\gamma_v,o_v)}$ for each $v \in \{v_e, v^e\}$. More precisely, we use the following choice.

I. Let $\mathbb{R}\Sigma_v \neq \emptyset$ for $v \in \{v_e, v^e\}$ and let $\mathrm{Fix}(\sigma) \neq \emptyset$. If $|F^{\mathbb{R}}_\gamma(v_e)| \geq 3$ (resp. $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$), then we specify an isomorphism $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ (resp. $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$) by mapping three consecutive special points as follows: If $|F^{\mathbb{R}}_\gamma(v_e)| > 3$ and the special points p_{f_e} and p_n are not consecutive, then $\Phi_{v_e} : (p_{i_{l-1}}, p_n, p_{i_1}) \mapsto (1,\infty,0)$. If $|F^{\mathbb{R}}_\gamma(v_e)| \geq 3$ and the special points p_{f_e} and p_n are consecutive and

$$\begin{array}{llll} \{f_e\} < \{n\} < \{i_1\}, & \Longrightarrow & \Phi_{v_e} : (p_{f_e}, p_n, p_{i_1}) & \mapsto \quad (1,\infty,0), \\ \{i_{l-1}\} < \{n\} < \{f_e\}, & \Longrightarrow & \Phi_{v_e} : (p_{i_{l-1}}, p_n, p_{f_e}) & \mapsto \quad (1,\infty,0). \end{array}$$

For $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$, $\Phi_{v^e} : (p_{i_{q+1}}, p_{i_{q+r}}, p_{f^e}) \mapsto (1,\infty,0)$.

If $|F^{\mathbb{R}}_\gamma(v_e)| < 3$ (resp. $|F^{\mathbb{R}}_\gamma(v^e)| < 3$), then in addition to $p_n \mapsto \infty$ (resp. $p_{f^e} \mapsto 0$), we pick the maximal element α in $F^+_\gamma(v_e)$ (resp. $F^+_\gamma(v^e)$) and map the pair of conjugate labeled points $(p_\alpha, p_{\overline{\alpha}})$ to $(\sqrt{-1}, -\sqrt{-1})$.

II. Let $\mathbb{R}\Sigma_v \neq \emptyset$ for $v \in \{v_e, v^e\}$ and let $\mathrm{Fix}(\sigma) = \emptyset$. We specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping the pair of conjugate labeled points $(p_\alpha, p_{\overline{\alpha}})$ to $(\sqrt{-1}, -\sqrt{-1})$ for the maximal element α in $F^+_\gamma(v)$, and $p_{f_e} \mapsto 0$ (resp. $p_{f^e} \mapsto 0$).

III. Let $\mathbb{R}\Sigma$ be an isolated real node. We pick a maximal element α_{k-1} in $F^+_\gamma(v_e) \setminus \{n\}$ and specify isomorphisms $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ and $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$ by mapping the special points $(p_{f_e}, p_{\alpha_{k-1}}, p_n)$ and, respectively, $(p_{f^e}, p_{\overline{\alpha}_{k-1}}, p_{\overline{n}})$ to $(0, \sqrt{-1}/2, \sqrt{-1})$.

By using the o-planar structure

$$o_v = \begin{cases} \{\mathbb{R}\Sigma_v \neq \emptyset; F^{\pm}_\gamma(v); F^{\mathbb{R}}_\gamma(v) = \{\{f_1\} < \cdots < \{f_{l_v}\}\}\} & \text{for case } \mathbf{I}, \\ \{\mathbb{R}\Sigma_v \neq \emptyset; F^{\pm}_\gamma(v); F^{\mathbb{R}}_\gamma(v) = \emptyset\} & \text{for case } \mathbf{II}, \end{cases}$$

of γ_v for each $v \in \{v_e, v^e\}$, arrange the coordinates of the special points in the following order

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}}, x_{i_1}, \cdots, x_{i_{l_v}}),$$

where $\alpha_* \in F_\gamma^+(v)$. We fix special points as in (**I**) and (**II**), and apply (8) to introduce top-dimensional differential forms $\Omega_{(\gamma_{v^e}, o_{v^e})}$ and $\Omega_{(\gamma_{v_e}, o_{v_e})}$ on $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$ (note that the resulting forms do not depend on the order of z-coordinates). In the case (**III**), there are no real special points, so we may get a top-dimensional differential form $\Omega_{(\gamma,o)}$ on $C_{(\gamma,o)}$ via choosing the vertex $v \in V_\gamma^+$ with ordering arbitrarily the z-coordinates

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}})$$

where $F_\gamma^+ = \{\alpha_1, \cdots, \alpha_{k_v}\}$.

In such a way, we produce well-defined orientations $[\Omega_{(\gamma_{v^e}, o_{v^e})}]$ and $[\Omega_{(\gamma_{v_e}, o_{v_e})}]$ of, $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$ respectively, and finally get an orientation on $C_{(\gamma,o)}$ determined by

$$\begin{array}{ll} [\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}] & \text{when } V_\gamma = V_\gamma^{\mathbb{R}} \\ [\Omega_{(\gamma,o)}] & \text{when } V_\gamma^{\mathbb{R}} = \emptyset \text{ and } v \in V_\gamma^+. \end{array}$$

5.3. Induced orientations on codimension one strata

Let $C_{(\tau,u)}$ be a top dimensional stratum, and let $C_{(\gamma,\hat{u})}$ be a codimension one stratum contained in the boundary of $\overline{C}_{(\tau,u)}$. We lift the u-planar structures $u, \hat{u}$ to o-planar representatives $o, \hat{o} = \delta(o)$ such that (τ, o) is obtained by contracting the edge of $(\gamma, \delta(o))$ (see Prop. 4.7). Then we pick a point $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\delta(o))}$ and consider a tubular neighborhood $\mathbb{R}V \times [0, \epsilon[$ of $(\Sigma^o, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ as in Section 4.5.

The orientation $[\omega_{(\tau,o)}]$, introduced in 5.1, induces some orientation on $C_{(\gamma,\delta(o))}$: The outward normal direction of $\overline{C}_{(\tau,o)}$ on $\mathbb{R}V \times \{0\} \subset C_{(\gamma,\delta(o))}$ is $-\partial_t$, where t is the standard coordinate on $[0, \epsilon[\subset \mathbb{R}$. Therefore a differential form $\omega_{(\gamma,\delta(o))}$ defines the induced orientation, if and only if

$$-dt \wedge \omega_{(\gamma,\delta(o))} = \Theta\, \omega_{(\tau,o)} \tag{9}$$

with $\Theta > 0$ at each point of $\mathbb{R}V \times]0, \epsilon[$.

In what follows, we compare the induced orientation $[\omega_{(\gamma,\delta(o))}]$ with $[\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$.

5.3.1. Case I: $\mathrm{Fix}(\sigma) \neq \emptyset$.

Lemma 5.1. *Let $(\gamma, \delta(o))$ be an o-planar tree as above, and let $|F_\gamma^{\mathbb{R}}| = l + 2 > 3$, and $|F_\gamma^{\mathbb{R}}(v^e)| = r + 1$. Then,*

$$[\omega_{(\gamma,\delta(o))}] = (-1)^{\aleph}\ [\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$$

where the values of $\aleph$ for separate cases are given in the following table.

ℵ	$l-r\geq 3$	$l-r=2$		$l-r=1$
		$\{i_1\}<\{f_e\}<\{n\}$	$\{f_e\}<\{i_{l-1}\}<\{n\}$	
$r\geq 2$	$(q+1)(r+1)$	0	$l+1$	$l+1$
$r=1$	1	1	1	1
$r=0$	$q+1$	0	0	0

Here, the third and fourth columns correspond to two possible cyclic orderings of $F^{\mathbb{R}}_\gamma(v_e)$ *for* $|F^{\mathbb{R}}_\gamma(v_e)|=3$ *in Case* **I** *of Section 5.2.*

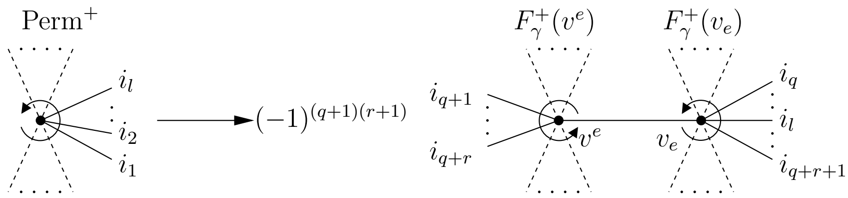

FIGURE 4. Codimension 1 boundaries of $\overline{C}_{(\tau,\delta(o))}$ where $l-r\geq 3$ & $r\geq 2$.

Proof. We will prove the statement only for the special case of $l-r\geq 3$, $r\geq 2$. The calculations for other cases are almost identical.

Let $(\Sigma^o,\mathbf{p}^o)\in C_{(\gamma,\delta(o))}$. We set $\Sigma^o_{v_e}$ to be $\{w=0\}$ and $\Sigma^o_{v^e}$ to be $\{z-x_{f_e}=0\}$. According to the convention in 5.2, the consecutive special points $(p_{i_{l-1}},p_n,p_{i_1})$ (resp. $(p_{i_{q+1}},p_{i_{q+r}},p_{f^e})$) on the component Σ_{v_e} (resp. Σ_{v^e}) are fixed at $(1,\infty,0)$. As shown in the proof of Proposition 4.7, a tubular neighborhood $\mathbb{R}V\times[0,\epsilon[$ of $(\Sigma^o,\mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ can be given by the family $\{(z-x_{f_e})\cdot w+t=0\mid t\in[0,\epsilon[\}$ with labeled points $p_{i_1}=(0,t/x_{f_e})$, $p_{i_{q+1}}=(x_{f_e}-t,1)$, $p_{i_{q+r}}=(x_{f_e},\infty)$, $p_{i_{l-1}}=(1,-t/(1-x_{f_e}))$, $p_n=(\infty,0)$, $p_{i_*}=(x_{i_*},-t/(x_{i_*}-x_{f_e}))$ for $i_*\in F^{\mathbb{R}}_\gamma\backslash\{i_1,i_{q+1},i_{q+r},i_{l-1},n\}$ and $p_\alpha=(z_\alpha,-t/(z_\alpha-x_{f_e}))$ for $\alpha\in F^+_\gamma$.

We first consider the following subcase: the special points p_{f_e} and p_n are not consecutive. According to 5.2, the differential forms $\Omega_{(\gamma_{v_e},o_{v_e})}$ and $\Omega_{(\gamma_{v^e},o_{v^e})}$ of this case are as follows:

$$\begin{aligned}
\Omega_{(\gamma_{v_e},o_{v_e})} &= \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma(v_e)|}\bigwedge_{\alpha\in F^+_\gamma(v_e)} dz_\alpha\wedge d\overline{z}_\alpha\wedge\\
&\quad dx_{i_2}\wedge\cdots\wedge dx_{i_q}\wedge dx_{f_e}\wedge\widehat{dx_{i_{q+1}}}\wedge\widehat{\cdots}\wedge\widehat{dx_{i_{q+r}}}\wedge dx_{i_{q+r+1}}\wedge\cdots\wedge dx_{i_{l-2}},\\
\Omega_{(\gamma_{v^e},o_{v^e})} &= \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma(v^e)|}\bigwedge_{\beta\in F^+_\gamma(v^e)} dw_\beta\wedge d\overline{w}_\beta\bigwedge dy_{i_{q+2}}\wedge\cdots\wedge dy_{i_{q+r-1}}
\end{aligned}$$

By using the identities $w_\beta = -t/(z_\beta - x_{f_e})$ for $\beta \in F^+_\gamma(v^e)$ and $y_i = -t/(x_i - x_{f_e})$ for $i \in F^{\mathbb{R}}_\gamma(v^e)$, we obtain the following equalities:

$$\begin{array}{ll} dt = -dx_{i_{q+1}} + dx_{f_e}, \quad dx_{f_e} = dx_{i_{q+r}}, & \\ dw_\beta = -\frac{dt}{z_\beta - x_{f_e}} + \frac{tdz_\beta}{(z_\beta - x_{f_e})^2} - \frac{tdx_{f_e}}{(z_\beta - x_{f_e})^2} & \text{for } \beta \in F^\pm_\gamma(v^e), \\ dy_i = -\frac{dt}{x_i - x_{f_e}} + \frac{tdx_i}{(x_i - x_{f_e})^2} - \frac{tdx_{f_e}}{(x_i - x_{f_e})^2} & \text{for } i = q+2, \cdots, q+r-1. \end{array}$$

These identities imply that $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}$ is equal to

$$(-1)^{(r-1)(q-1)} \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma|} \Theta \bigwedge_{\alpha \in F^+_\gamma} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{l-2}$$

where $\Theta = \prod_{\beta \in F^+_\gamma(v^e)} t(z_\beta - x_{f_e})^{-2} \prod_{i_{q+2}, \cdots, i_{q+r-1}} t(x_i - x_{f_e})^{-2}$. Since $\Theta > 0$, the orientation defined by $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v^e}, o_{v^e})}$ is equal to $(-1)^{\aleph}[\omega_{(\tau,o)}]$.

We now consider the cases $\{f_e\} < \{n\} < \{i_1\}$ (i.e., $q + r = l - 1$) and $\{i_{l-1}\} < \{n\} < \{f_e\}$ (i.e., $q = 0$). According to 5.2, the differential forms $\Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}$ are equal to

$$\left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma|} \begin{array}{l} \left(\bigwedge_{\beta \in F^+_\gamma(v^e)} dz_\beta \wedge d\overline{z}_\beta \bigwedge dy_{i_{q+2}} \wedge \cdots \wedge dy_{i_{l-2}}\right) \\ \wedge \left(\bigwedge_{\alpha \in F^+_\gamma(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}\right) \end{array}$$

when $q + r = l - 1$, and

$$\left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma|} \begin{array}{l} \left(\bigwedge_{\beta \in F^+_\gamma(v^e)} dz_\beta \wedge d\overline{z}_\beta \bigwedge dy_{i_2} \wedge \cdots \wedge dy_{i_{r-1}}\right) \\ \wedge \left(\bigwedge_{\alpha \in F^+_\gamma(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_{r+1}} \wedge \cdots \wedge dx_{i_{l-2}}\right) \end{array}$$

when $q = 0$. The equation $(z - x_{f_e}) \cdot w + t = 0$ implies the following equalities:

$$\begin{array}{ll} dt = -dx_{i_{q+1}}, \quad dx_{f_e} = dx_{i_{l-1}} & \text{when } q + r = l - 1 \\ dt = dx_{i_r}, \quad dx_{f_e} = dx_{i_{q+r}} & \text{when } q = 0, \end{array}$$

and

$$\begin{array}{ll} dw_\beta = -\frac{dt}{z_\beta - x_{f_e}} + \frac{tdz_\beta}{(z_\beta - x_{f_e})^2} - \frac{tdx_{f_e}}{(z_\beta - x_{f_e})^2} & \text{for } \beta \in F^\pm_\gamma(v^e), \\ dy_i = -\frac{dt}{x_i - x_{f_e}} + \frac{tdx_i}{(x_i - x_{f_e})^2} - \frac{tdx_{f_e}}{(x_i - x_{f_e})^2} & \text{for } i = i_{q+2}, \cdots, i_{q+r-1}. \end{array}$$

By using these identities we obtain that $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v^e}, o_{v^e})}$ is equal to

$$(-1)^{(q-1)(l-q-2)} \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma|} \Theta \bigwedge_{\alpha \in F^+_\gamma} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{i_{l-2}}$$

when $q+r=l-1$, and

$$(-1)^{(r-1)}\left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|}\Theta\bigwedge_{\alpha\in F_\gamma^+}dz_\alpha\wedge d\overline{z}_\alpha\bigwedge dx_{i_2}\wedge\cdots\wedge dx_{i_{l-2}}$$

when $q=0$. Since $\Theta=\prod_{\beta\in F_\gamma^+(v^e)}t(z_\beta-x_{f_e})^{-2}\prod_{i_{q+2},\cdots,i_{q+r-1}}t(x_i-x_{f_e})^{-2}>0$, the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by $[\omega_{(\tau,o)}]$ is equal to

$$(-1)^{\aleph}\ \left[\Omega_{(\gamma_{v^e},o_{v^e})}\wedge\Omega_{(\gamma_{v_e},o_{v_e})}\right]=$$
$$\begin{cases}(-1)^{(q+1)(r-1)}\left[\Omega_{(\gamma_{v^e},o_{v^e})}\wedge\Omega_{(\gamma_{v_e},o_{v_e})}\right] & \text{when } q+r=l-1,\\ (-1)^{(r-1)}\left[\Omega_{(\gamma_{v^e},o_{v^e})}\wedge\Omega_{(\gamma_{v_e},o_{v_e})}\right] & \text{when } q=0.\end{cases}$$

□

5.3.2. Case II. Fix$(\sigma)=\emptyset$.

The different cases for boundaries of $C_{(\tau,o)}$ are treated separately. The proofs are essentially the same as the proof of Lemma 5.1.
Subcase $\mathbb{R}\Sigma\neq\emptyset$. Let (τ,o) be a one-vertex o-planar tree with $\mathbb{R}\Sigma\neq\emptyset$, and let $[\omega_{(\tau,o)}]$ be, in accordance with 5.1, the orientation of $C_{(\tau,o)}$ defined by the differential form

$$\omega_{(\tau,o)}:=(\frac{\sqrt{-1}}{2})^{k-2}\bigwedge_{\alpha_*\in\mathrm{Perm}^+\setminus(\mathrm{Perm}^+\quad\{k-1,k,2k-1,2k\})}dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*}\bigwedge d\lambda \qquad (10)$$

(which is given by the coordinates in **(C)** of 4.1.2). Here $\lambda=\Im(z_{\alpha'})$ and $\alpha'\in\{k-1,2k-1\}\bigcap\mathrm{Perm}^+$.

Lemma 5.2. *Let $(\gamma,\delta(o))$ be a two-vertex o-planar tree, and let the corresponding strata $C_{(\gamma,\delta(o))}$ be contained in the boundary of $\overline{C}_{(\tau,o)}$.*
(i) If $V_\gamma^{\mathbb{R}}=V_\gamma$, then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $-[\Omega_{(\gamma_{v^e},o_{v^e})}]\wedge[\Omega_{(\gamma_{v_e},o_{v_e})}]$.
(ii) If $V_\gamma^{\mathbb{R}}=\emptyset$, then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $[\Omega_{(\gamma,\delta(o))}]$.

Subcase $\mathbb{R}\Sigma=\emptyset$. Let (τ,o) be a one-vertex o-planar tree where $o=\{\mathbb{R}\Sigma=\emptyset\}$ and let $[\omega_{(\tau,o)}]$ be, in accordance with 5.1, the orientation of $C_{(\tau,o)}$ defined by

$$\omega_{(\tau,o)}:=-(\frac{\sqrt{-1}}{2})^{k-2}\bigwedge_{\alpha_*\in\{1,\cdots,k-2\}}dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*}\bigwedge d\lambda \qquad (11)$$

(which is given by the coordinates in **(D)** of 4.1.2). Here $\lambda=\Im(z_{k-1})$.

Lemma 5.3. *Let $(\gamma,\delta(o))$ be a two-vertex o-planar tree where $V_\gamma^{\mathbb{R}}=\emptyset$, and let $C_{(\gamma,\delta(o))}$ be contained in the boundary of strata $\overline{C}_{(\tau,o)}$ given above. Then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $(-1)^{\aleph}[\Omega_{(\gamma,\delta(o))}]$ where $\aleph$ is given by $|\{1,\cdots,k-1\}\bigcap F_\gamma^-|+1$.*

5.4. Conventions

Let $(\tau, o_\star)$ be the one-vertex o-planar tree where the o-planar structure $o_\star$ is given by $\mathbb{R}\Sigma \neq \emptyset$, $F_\tau^+ = \{1, 2, \cdots, k\}$, $F_\tau^- = \{k+1, \cdots, 2k\}$, and $F_\tau^{\mathbb{R}} = \{2k+1\} < \{2k+2\} < \cdots < \{2k+l := n\}$. All the other o-planar structures with $\mathbb{R}\Sigma \neq \emptyset$ on τ are obtained as follows.

Let $\varrho \in S_n$ be a permutation which commutes with σ and, if $l > 0$, preserves n. It determines an o-planar structure given by $\varrho(o_\star) = \{\mathbb{R}\Sigma \neq \emptyset; \varrho(\text{Perm}^\pm);\ \text{Fix}(\sigma) = \{\{\varrho(2k+1)\} < \cdots < \{\varrho(2k+l-1)\} < \{\varrho(n) = n\}\}$. The parity of ϱ depends only on $o = \varrho(o_\star)$ and we call it *parity* $|o|$ of $o = \varrho(o_\star)$.

5.4.1. Convention of orientations

We fix an orientation for each top-dimensional stratum as follows.

a. Case $\mathbb{R}\Sigma \neq \emptyset$. First, we select o-planar representatives for each one-vertex u-planar tree with $\mathbb{R}\Sigma \neq \emptyset$ as follows:

(1) if $l \geq 3$, we choose the representative (τ, o) of (τ, u) for which $\{2k+1\} < \{n-1\} < \{n\}$;

(2) if $l < 3$, we choose the representative (τ, o) of (τ, u) such that $k \in \text{Perm}^+$.

We denote the set of these o-planar representatives of u-planar trees by $\mathcal{U}Tree(\sigma)$, and select the orientation for $C_{(\tau,u)} = C_{(\tau,o)}$ with $C_{(\tau,o)} \in \mathcal{U}Tree(\sigma)$ to be

$$(-1)^{|o|}[\omega_{(\tau,o)}], \tag{12}$$

where $\omega_{(\tau,o)}$ is the form defined according to 5.1 and $|o|$ is the parity introduced in 5.4.

b. Case $\mathbb{R}\Sigma = \emptyset$**.** Here, we choose the orientation defined by the form (11).

In what follows, if $\mathbb{R}\Sigma \neq \emptyset$ we denote the set of flags $\{2k+1, n-1, n\}$ (for $l \geq 3$ case) and $\{k, 2k, n\}$ (for $l < 3$ case) by $\mathfrak{F}$.

5.5. Adjacent top-dimensional strata with $\mathbb{R}\Sigma \neq \emptyset$

Let $C_{(\tau,u_i)}, i = 1, 2$, be a pair of adjacent top-dimensional strata with $\mathbb{R}\Sigma \neq \emptyset$, and $C_{(\gamma,u)}$ be their common codimension one boundary stratum. Let (τ, o_i) be the o-planar representatives of (τ, u_i) given in 5.4.1. Consider the pair of o-planar representatives $(\gamma, \delta(o_i))$ of (γ, u) which respectively give (τ, o_i) after contracting their edges.

Lemma 5.4. *The o-planar tree $(\gamma, \delta(o_1))$ is obtained by reversing the o-planar structure $\delta(o_2)_v$ of $(\gamma, \delta(o_1))$ at vertex v where $|F_\gamma(v) \bigcap \mathfrak{F}| \leq 1$.*

Proof. Obviously, $(\gamma, \delta(o_1))$ can be obtained from $(\gamma, \delta(o_2))$ by reversing the o-planar structures at one or both its vertices v_e, v^e. If we reverse the o-planar structure of $(\gamma, \delta(o_2))$ at the vertex v such that $|F_\gamma(v) \bigcap \mathfrak{F}| > 1$, or at both of its vertices v_e and v^e, then the resulting o-planar tree will not be an element of $\mathcal{U}Tree(\sigma)$ after contracting its edge: reversing the o-planar structure at the vertex v with $|F_\gamma(v) \bigcap \mathfrak{F}| > 1$, or at both of the vertices reverses cyclic order of the elements $\{2k+1, n-1, n\}$ when $l \geq 3$, and moves k from Perm^+ to Perm^- when $l < 3$ (see Figure 5). $\square$

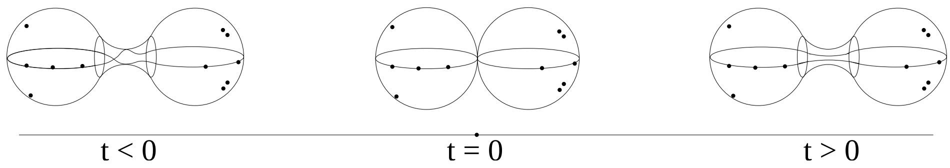

FIGURE 5. Two possible deformation of a real double point.

For a pair of two-vertex o-planar trees $(\gamma, \delta(o_i))$ as above, we calculate the differences of parities as follows.

Lemma 5.5. *Let $(\gamma, \delta(o_i)), i = 1, 2$ be a pair of o-planar trees as above. Let $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, and let o-planar structures at the vertices v_e and v^e be*

$$\begin{aligned} \delta(o_1)_{v_e} &= \left\{ \begin{array}{l} \mathbb{R}\Sigma_{v_e} \neq \emptyset; F_\gamma^{\pm}(v_e); F_\gamma^{\mathbb{R}}(v_e) = \{\{i_1\} < \cdots < \{i_q\} < \{f_e\} < \\ < \{i_{q+r+1}\} < \cdots < \{i_{l-1}\} < \{n\}\} \end{array} \right\} \\ \delta(o_2)_{v^e} &= \{\mathbb{R}\Sigma_{v^e} \neq \emptyset; F_\gamma^{\pm}(v^e); F_\gamma^{\mathbb{R}}(v^e) = \{\{i_{q+1}\} < \cdots < \{i_{q+r}\} < \{f^e\}\}\}. \end{aligned}$$

Let v be the vertex such that $|F_\gamma(v) \bigcap \mathfrak{F}| \leq 1$. Then,

$$|o_1| - |o_2| = \begin{cases} |F_\gamma^+(v^e)| + \frac{r(r-1)}{2} & \text{if } v = v^e \\ |F_\gamma^+(v_e)| + qr + rs + qs + \frac{s(s-1)}{2} + \frac{q(q-1)}{2} & \text{if } v = v_e \text{ and } |F_\gamma^{\mathbb{R}}(v_e)| > 3, \\ |F_\gamma^+(v_e)| + |F_\gamma^{\mathbb{R}}(v^e)| - 1 & \text{if } v = v_e \text{ and } |F_\gamma^{\mathbb{R}}(v_e)| = 3, \\ |F_\gamma^+(v_e)| & \text{if } v = v_e \text{ and } |F_\gamma^{\mathbb{R}}(v_e)| = 2. \end{cases}$$

Here, $r = |F_\gamma^{\mathbb{R}}(v^e)| - 1$ and $s = |F_\gamma^{\mathbb{R}}(v_e)| - q - 2$.

In Section 5.2, we have introduced differential forms $\Omega_{(\gamma_v, \delta(o_i)_v)}$ for each $v \in V_\gamma$. When we reverse the o-planar structure at the vertex v, the differential forms $\Omega_{(\gamma_v, \delta(o_2)_v)}$, $\Omega_{(\gamma_v, \delta(o_1)_v)}$ become related as follows.

Lemma 5.6. *Let $(\gamma, \delta(o_i)), i = 1, 2$ be two-vertex o-planar trees as above. Then,*

$$\Omega_{(\gamma_v, \delta(o_1)_v)} = (-1)^{\mu(v)}\ \Omega_{(\gamma_v, \delta(o_2)_v)},$$

where

$$\mu(v) = |F_\gamma^+(v)| + \frac{(|F_\gamma^{\mathbb{R}}(v)| - 2)(|F_\gamma^{\mathbb{R}}(v)| - 3)}{2}.$$

Lemmata 5.5 and 5.6 follow from straightforward calculations.

5.6. The first Stiefel-Whitney class

This section is devoted to the proof of the following theorem.

Theorem 5.7. *(i) For* $\mathrm{Fix}(\sigma) \neq \emptyset$*, the Poincaré dual of the first Stiefel-Whitney class of* $\mathbb{R}\overline{M}_{(2k,l)}$ *is*

$$[w_1] \;=\; \sum_{(\gamma,u)} [\overline{C}_{(\gamma,u)}] \;=\; \sum_{\gamma} [\mathbb{R}\overline{D}_\gamma] \quad \mod 2,$$

where the both sums are taken over all two-vertex trees such that

- $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$ *and* $|v^e| = 0 \mod 2$*, or*
- $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$*,* $|F_\gamma^{\mathbb{R}}(v_e)| \neq 3$ *and* $|v_e|(|v^e|-1) = 0 \mod 2$*, or*
- $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$*,* $|F_\gamma^{\mathbb{R}}(v_e)| = 3$ *and* $|F_\gamma^{\mathbb{R}}(v^e)| = 1$*,*

and, in the first sum, in addition over all u-planar structures on γ*.*

(ii) For $\mathrm{Fix}(\sigma) = \emptyset$*, the Poincaré dual of the first Stiefel-Whitney class of* $\mathbb{R}\overline{M}_{(2k,0)}$ *vanishes.*

Proof. Fix an orientation for each top-dimensional stratum as in 5.4.1. The orientation $(-1)^{|o|}[\omega_{(\tau,o)}]$ of a top-dimensional stratum $C_{(\tau,o)}$ induces some orientation of each codimension one stratum $C_{(\gamma,\delta(o))}$ (and $C_{(\gamma,\hat{o})}$) contained in the boundary of $\overline{C}_{(\tau,o)}$. The induced orientations $(-1)^{|o|}[\omega_{(\gamma,\delta(o))}]$ and $(-1)^{|o|}[\omega_{(\gamma,\hat{o})}]$ are determined in Lemmata 5.1, 5.2 and 5.3, and they give (relative) fundamental cycles $[\overline{C}_{(\gamma,\delta(o))}]$ and $[\overline{C}_{(\gamma,\hat{o})}]$ of the codimension one strata $\overline{C}_{(\gamma,\delta(o))}$ and $\overline{C}_{(\gamma,\hat{o})}$ respectively.

The Poincaré dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ is given by

$$[w_1] = \begin{cases} \frac{1}{2}\sum_{(\tau,u)} \left(\sum_{(\gamma,\delta(o))}[\overline{C}_{(\gamma,\delta(o))}]\right) \mod 2, & \text{when } l > 0, \\ \frac{1}{2}\sum_{(\tau,u)} \left(\sum_{(\gamma,\hat{o})}[\overline{C}_{(\gamma,\hat{o})}]\right) \mod 2, & \text{when } l = 0, \end{cases} \tag{13}$$

where the external summation runs over all one-vertex u-planar trees (τ,u) and the internal one over all codimension one strata of $\overline{C}_{(\tau,o)}$ for the one-vertex o-planar tree (τ,o) which represents (τ,u) in accordance with 5.4.1. Indeed, the sum (13) detects where the orientation on $\mathbb{R}M_{(2k,l)}$ can not be extended to $\mathbb{R}\overline{M}_{(2k,l)}$.

We prove the theorem by evaluating (13).

Case $\mathrm{Fix}(\sigma) \neq \emptyset$**.** Let $C_{(\tau,o_i)}, i = 1,2$ be a pair of adjacent top-dimensional strata, and $C_{(\gamma,\delta(o_i))} \subset \overline{C}_{(\tau,o_i)}$ be their common codimension one boundary stratum. We calculate $\left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \left[\overline{C}_{(\gamma,\delta(o_2))}\right]$ as follows. According to 5.4.1, the strata $C_{(\tau,o_i)}$ are oriented by $(-1)^{|o_i|}[\omega_{(\tau,o_i)}]$, and these orientations induce the orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ on $C_{(\gamma,\delta(o_i))}$. The induced orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ are given by

$$(-1)^{|o_i|+\aleph_i}[\Omega_{(\gamma_{v^e},\delta(o_i)_{v^e})} \wedge \Omega_{(\gamma_{v_e},\delta(o_i)_{v_e})}]$$

in Lemmata 5.1 and 5.2 according to the convention introduced in Section 5.2. We denote by v be the vertex such that $|F_\gamma(v) \bigcap \mathfrak{F}| \leq 1$ as in Section 5.5, and compare the induced orientations by calculating

$$\Pi(o_1,o_2) = (|o_1| + \aleph_1) - (|o_2| + \aleph_2) - \mu(v)$$

for each of the following three subcases.

First, assume that $|F_\gamma(v^e)\bigcap \mathfrak{F}| \leq 1$. In this subcase, the o-planar structure is reversed at the vertex $v = v^e$. Therefore, $\aleph_1 = \aleph_2$ according to Lemma 5.1. Finally, by applying Lemmata 5.5 and 5.6 and using relation $r = |F^{\mathbb{R}}_\gamma(v^e)| - 1$ we obtain

$$\begin{aligned} \Pi(o_1,o_2) = |o_1| - |o_2| - \mu(v^e) &= \frac{r(r-1)}{2} - \frac{(|F^{\mathbb{R}}_\gamma(v^e)|-2)(|F^{\mathbb{R}}_\gamma(v^e)|-3)}{2} \\ &= |F^{\mathbb{R}}_\gamma(v^e)| - 2 \\ &= |v^e| \mod 2. \end{aligned}$$

The latter equality follows from the fact that $|F^{\mathbb{R}}_\gamma(v)| = |v| \mod 2$.

Second, assume that $|F_\gamma(v_e)\bigcap \mathfrak{F}| \leq 1$ and $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$. In this subcase, the o-planar structure is reversed at the vertex $v = v_e$. Since $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$, once more $\aleph_1 = \aleph_2$ according to the Lemma 5.1. Finally, by applying Lemmata 5.5 and 5.6 and using relation $|F^{\mathbb{R}}_\gamma(v_e)| = q + s + 2$, we obtain

$$\begin{aligned} \Pi(o_1,o_2) &= qr + rs + qs + \frac{s(s-1)}{2} + \frac{q(q-1)}{2} - \frac{(q+s)(q+s-1)}{2} \\ &= r(q+s), \\ &= (|F^{\mathbb{R}}_\gamma(v^e)|-1)(|F^{\mathbb{R}}_\gamma(v_e)|-2), \\ &= |v_e|(|v^e|-1) \mod 2 \end{aligned}$$

when $|F^{\mathbb{R}}_\gamma(v_e)| > 3$, and

$$\begin{aligned} \Pi(o_1,o_2) &= 2|F^{+}_\gamma(v_e)| = 0 \mod 2 \\ &= |v_e|(|v^e|-1) \mod 2 \end{aligned}$$

when $|F^{\mathbb{R}}_\gamma(v_e)| = 2$.

Third, we consider $|F_\gamma(v_e)\bigcap \mathfrak{F}| \leq 1$ and $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ case. In this subcase, the o-planar structure is reversed at the vertex v_e. Hence, $\aleph_1 = \aleph_2$ whenever $|F^{\mathbb{R}}_\gamma(v^e)| = 1, 2$, and $\aleph_1 - \aleph_2$ is $\pm(l+1) = \pm(|F^{\mathbb{R}}_\gamma(v^e)| + 2)$ whenever $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$. Finally, by applying Lemmata 5.5 and 5.6, we obtain

$$\Pi(o_1,o_2) = \begin{cases} |F^{\mathbb{R}}_\gamma(v^e)| - 1 \pm (|F^{\mathbb{R}}_\gamma(v^e)| + 2) & = 1 \mod 2, \quad \text{when } |F^{\mathbb{R}}_\gamma(v^e)| \geq 3, \\ |F^{\mathbb{R}}_\gamma(v^e)| - 1 & = 1 \mod 2, \quad \text{when } |F^{\mathbb{R}}_\gamma(v^e)| = 2, \\ |F^{\mathbb{R}}_\gamma(v^e)| - 1 & = 0 \mod 2, \quad \text{when } |F^{\mathbb{R}}_\gamma(v^e)| = 1, \end{cases}$$

The induced orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ are the same if and only if $\Pi(o_1,o_2) = 0 \mod 2$. Hence, we have

$$\left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \left[\overline{C}_{(\gamma,\delta(o_2))}\right] = \frac{1 + (-1)^{\Pi(o_1,o_2)}}{2} \left[\overline{C}_{(\gamma,\delta(o_1))}\right].$$

The sum $([\overline{C}_{(\gamma,\delta(o_1))}]+[\overline{C}_{(\gamma,\delta(o_2))}])/2$ gives us the fundamental cycle $[\overline{C}_{(\gamma,\delta(o_1))}]$ when $(-1)^{|o_1|}[\omega_{(\gamma,\delta(o_1))}] = (-1)^{|o_2|}[\omega_{(\gamma,\delta(o_2))}]$, and it turns to zero otherwise. Finally, as it follows from the above case-by-case calculations of $\Pi(o_1,o_2)$, the fundamental class of a codimension one strata $\overline{C}_{(\gamma,\delta(o_1))}$ is involved in $[w_1]$ if and only if one of the three conditions given in Theorem are verified. It gives the first expression for $[w_1]$ given in Theorem. Since in this first expression the sum is taken over all u-planar structures on γ, it can be shorten to the sum of the fundamental classes of $\mathbb{R}D_\gamma$.

Case $\mathrm{Fix}(\sigma)=\emptyset$**.** Let $C_{(\tau,u_i)}, i=1,2$, be a pair of adjacent top-dimensional strata and $C_{(\gamma,u)}$ be their common codimension one boundary stratum. Let (τ,o_i) be the o-planar representatives of (τ,u_i) given in 5.4.1. Here, we have to consider two subcases: (i) $C_{(\gamma,u)}$ is a stratum of real curves with two real components (i.e., $|V_\gamma|=|V_\gamma^{\mathbb{R}}|=2$), and (ii) $C_{(\gamma,u)}$ is a stratum of real curves with two complex conjugated components (i.e., $|V_\gamma|=2$ and $|V_\gamma^{\mathbb{R}}|=0$).

(i) Consider the pair of o-planar representatives $(\gamma,\delta(o_i))$ of (γ,u) which respectively give (τ,o_i) after contracting the edges and compare their o-planar structure. Since the both tails n and $\sigma(n)$ are in $F_\gamma(v_e)$, the o-planar structure is reversed at the vertex v^e. Therefore, $\aleph_1=\aleph_2$ according to the Lemma 5.1. Finally, by applying Lemmata 5.5 and 5.6, we obtain

$$\Pi(o_1,o_2) = 2|F_\gamma^+(v^e)|+1 = 1 \mod 2.$$

In other words, $\left[\overline{C}_{(\gamma,\delta(o_1))}\right]+\left[\overline{C}_{(\gamma,\delta(o_2))}\right]=0$ for this case.

(ii) Let $C_{(\tau,o_2)}$ be a stratum of real curves with empty real part, and let $(\gamma,\hat{o})$ be an o-planar representative of (γ,u).

The orientations of $C_{(\gamma,u)}$ induced by the orientations $(-1)^{|o_1|}[\omega_{(\tau,o_1)}]$ and $[\omega_{(\tau,o_2)}]$ of $C_{(\tau,o_1)}$ and $C_{(\tau,o_2)}$ are given in Lemmata 5.2 and 5.3. Namely, they are respectively given by the following differential forms

$$(-1)^{|o_1|} \bigwedge_{\alpha_*\in F_\gamma^+\setminus(F_\gamma^+\ \{k-1,k,2k-1,2k\})} dz_{\alpha_*}\wedge d\bar{z}_{\alpha_*},$$

$$(-1)^{\aleph} \bigwedge_{\alpha_*\in F_\gamma^+\setminus(F_\gamma^+\ \{k-1,k,2k-1,2k\})} dz_{\alpha_*}\wedge d\bar{z}_{\alpha_*},$$

where $|o_1| = |\{1,\cdots,k-1\}\bigcap F_\gamma^-|$ and $\aleph = |\{1,\cdots,k-1\}\bigcap F_\gamma^-|+1$. Therefore, the orientations induced from different sides are opposite and the sum $(-1)^{\aleph-1}\left[\overline{C}_{(\gamma,u)}\right] + (-1)^{\aleph}\left[\overline{C}_{(\gamma,u)}\right]$ vanishes for all such $(\gamma,\hat{o})$. □

5.6.1. Example

Due to Theorem 5.7, the Poincaré dual of the first Stiefel-Whitney class $[w_1]$ of $\mathbb{R}\overline{M}_{(0,5)}$ can be represented by $\sum_\gamma[\mathbb{R}\overline{D}_\gamma] = \sum_{(\gamma,u)}[\overline{C}_{(\gamma,u)}]$ where γ are 5-trees with a vertex v satisfying $|v|=4$ and $|F_\gamma(v)\bigcap\{1,4,5\}|=1$. These 5-trees are given in Figure 6a, and the union corresponding strata $\bigcup_\tau \mathbb{R}\overline{D}_\gamma$ is given the three exceptional divisors obtained by blowing up the three highlighted points in Figure 6b.

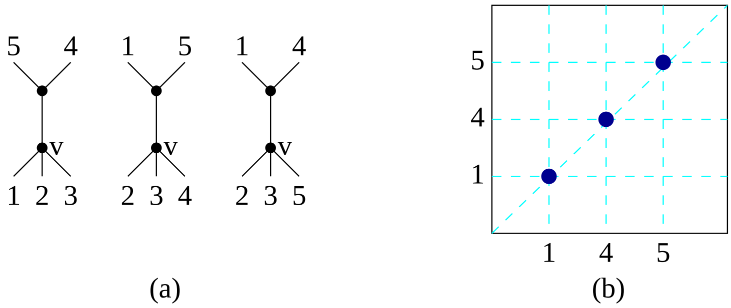

FIGURE 6. (a) The 5-trees of Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(0,5)}$ due Theorem 5.7, (b) The blown-up locus in $\mathbb{R}\overline{M}_{(0,5)}$

6. The orientation double covering of $\mathbb{R}\overline{M}_{(2k,l)}$

In Section 5.6, the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ is determined in terms of its strata. We have also proved that the moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ is orientable when $n = 4$ or $l = 0$. In this section, we give a combinatorial construction of orientation double covering for the rest of the cases i.e., $n > 4$ and $l > 0$. By observing the non-triviality of the orientation double cover in these cases, we prove that $\mathbb{R}\overline{M}_{(2k,l)}$ is not orientable.

6.1. Construction of orientation double covering

In Section 4.1.1, we have shown that the map $\widetilde{C}_{(2k,l)} \to \mathbb{R}M_{(2k,l)}$, which is identifying the reverse o-planar structures, is a trivial double covering. The disjoint union of closed strata $\overline{C}_{(2k,l)} = \bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$, where $|V_\tau| = 1$ and (τ, o) runs over all possible o-planar structures on τ, is a natural compactification of $\widetilde{C}_{(2k,l)}$.

To obtain the orientation double covering of $\mathbb{R}\overline{M}_{(2k,l)}$ we need to get rid of the codimension one strata by pairwise gluing them. We use the following simple recipe: for each pair $(\tau, o_i), = 1, 2,$ of one-vertex o-planar trees obtained by contracting the edge in a pair $(\tau, \delta(o_i)), i = 1, 2,$ of two-vertex o-planar trees with the same underlying tree such that $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, $v_e = \partial_\gamma(n)$, we glue $\overline{C}_{(\tau,o_i)}$ along $\overline{C}_{(\gamma,\delta(o_i))}, i = 1, 2,$ if

$\mathcal{A}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 1 \mod 2$,

$\mathcal{B}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 0 \mod 2$,

$\mathcal{C}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 1 \mod 2$,
$\mathcal{D}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 0 \mod 2$,
$\mathcal{E}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| = 3$ and $|F_\gamma^{\mathbb{R}}(v^e)| \neq 1$,
$\mathcal{F}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| = 3$ and $|F_\gamma^{\mathbb{R}}(v^e)| = 1$.

We denote by $\mathbb{R}\widetilde{M}_{(2k,l)}$ the resulting factor space.

Theorem 6.1. *$\mathbb{R}\widetilde{M}_{(2k,l)}$ is the orientation double cover of $\mathbb{R}\overline{M}_{(2k,l)}$.*

Proof. Let $\widetilde{M}$ be the orientation double covering of $\mathbb{R}\overline{M}_{(2k,l)}$. The points of $\widetilde{M}$ can be considered as points in $\mathbb{R}\overline{M}_{(2k,l)}$ with local orientation. On the other hand, by using opposite o-planar structures of a one-vertex τ we can determine orientations $(-1)^{|o|}[\omega_{(\tau,o)}]$ and $(-1)^{|\bar{o}|+l-1}[\omega_{(\tau,\bar{o})}]$ on $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ where $(\tau, o) \in \mathcal{U}Tree(\sigma)$. These orientations are opposite with respect to the identification of $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ by the canonical diffeomorphism $-\mathbb{I}$ introduced in Subsection 4.1.1. Hence, there is a natural continuous embedding $\widetilde{C}_{(2k,l)} = \bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)}(C_{(\tau,o)} \sqcup C_{(\tau,\bar{o})}) \to \widetilde{M}$. It extends to a surjective continuous map $\overline{C}_{(2k,l)} = \bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)}(\overline{C}_{(\tau,o)} \sqcup \overline{C}_{(\tau,\bar{o})}) \to \widetilde{M}$. Since $\overline{C}_{(2k,l)}$ is compact and $\widetilde{M}$ is Hausdorff, the orientation double covering $\widetilde{M}$ is a quotient space $\overline{C}_{(2k,l)}/R$ of $\overline{C}_{(2k,l)}$ under the equivalence relation R defined by the map $\overline{C}_{(2k,l)} \to \widetilde{M}$.

This equivalence relation is uniquely determined by its restriction to the codimension one faces of $\overline{C}_{(2k,l)}$, which cover the codimension one strata of $\mathbb{R}\overline{M}_{(2k,l)}$ under the composed map $\overline{C}_{(2k,l)} \to \widetilde{M} \to \mathbb{R}\overline{M}_{(2k,l)}$. On the other hand, the equivalence relation on the codimension one faces is determined by the first Stiefel-Whitney class: A partial section of the induced map $\overline{C}_{(2k,l)}/R \to \mathbb{R}\overline{M}_{(2k,l)}$ given by distinguished strata $\bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)} C_{(\tau,o)}$. Over a neighborhood of a codimension one stratum of $\mathbb{R}\overline{M}_{(2k,l)}$, a partial section extends to a section if this strata is not involved in the expression for the first Stiefel-Whitney class given in Theorem 5.7, and it should not extend, otherwise. Notice that the faces $\overline{C}_{(\tau,\delta(o_i))}$ considered in relations $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{E}$ are mapped onto the strata $\overline{C}_{(\tau,\delta(u))}$ which do not contribute to the expression $[w_1]$ given in Theorem 5.7, and the faces $\overline{C}_{(\tau,\delta(o_i))}$ in relations $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{F}$ are mapped onto the strata $\overline{C}_{(\tau,\delta(u))}$ which contribute to the expression $[w_1]$. There are four different faces $\overline{C}_{(\tau,\delta(o)_i)}, i = 1, \cdots, 4$ over each codimension one stratum $\overline{C}_{(\tau,\delta(u))}$. Lemma 5.4 determines the pairs $\overline{C}_{(\tau,\delta(o)_i)}$, $\overline{C}_{(\tau,\delta(o)_j)}$ to be glued to each other. □

Corollary 6.2. *The moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ is not orientable when $2k + l > 4$ and $l > 0$.*

Proof. Let $l \geq 3$, and (τ, o) be an o-planar structure with $\{2k+1\} < \{n-1\} < \{n\}$. It is clear that, we can produce any o-planar structure on τ with $\{2k+1\} < \{n-1\} < \{n\}$ by applying following operations consecutively:

- interchanging the order of two consecutive tails $\{i, i+1\}$ for $|\{i, i+1\} \bigcap \mathfrak{F}| \leq 1$ and $n \notin \{i, i+1\}$,
- swapping $j \in \mathrm{Perm}^+$ with $\bar{j} \in \mathrm{Perm}^-$ for $j \neq k, 2k$.

The one-vertex o-planar trees with $\{n-1\} < \{2k+1\} < \{n\}$ can be produced from the o-planar tree $(\tau, \bar{o})$ via same procedure.

Let $l = 1, 2$. Similarly, if we start with o-planar tree (τ, o) with $k \in \mathrm{Perm}^+$ ($k \in \mathrm{Perm}^-$), we can produce any o-planar structure on τ with $k \in \mathrm{Perm}^+$ ($k \in \mathrm{Perm}^-$) by swapping $j \in \mathrm{Perm}^+$ with $\bar{j} \in \mathrm{Perm}^-$ for $j \neq k, 2k$.

Note that, these operations correspond to passing from one top-dimensional stratum to another in $\mathbb{R}\widetilde{M}_{(2k,l)}$ through certain faces. These faces correspond to the one-edge o-planar trees $(\gamma, \delta(o)_i)$ with $F_\gamma(v^e) = \{i, i+1, f^e\}$ (resp. $F_\gamma(v^e) = \{j, \bar{j}, f^e\}$) which are faces glued according to the relations of type $\mathcal{A}$. Hence, any two top-dimensional strata in $\mathbb{R}\widetilde{M}_{(2k,l)}$ with same cyclic ordering of $\mathfrak{F}$ (resp. with k is in same set $\mathrm{Perm}^\pm$) can be connected through a path passing through these codimension faces $\overline{C}_{(\gamma, \delta(o)_i)}$. The quotient space $\overline{C}_{(2k,l)}/\mathcal{A}$ has two connected components since there are two possible cyclic orderings of $\mathfrak{F}$ when $l \geq 3$ (resp. two possibilities for $l = 1, 2$ case: $k \in \mathrm{Perm}^+$ and $k \in \mathrm{Perm}^-$).

The set of relations of type $\mathcal{B}$ is not empty when $2k + l > 4$ and $l > 0$. Moreover, the relation of type $\mathcal{B}$ reverses the cyclic ordering on $\mathfrak{F}$ (resp. moves k from $\mathrm{Perm}^\pm$ to $\mathrm{Perm}^\mp$). Hence, the faces glued according to the relations of type $\mathcal{B}$ connect two different components of $\overline{C}_{(2k,l)}/\mathcal{A}$. Therefore, the orientation double cover $\mathbb{R}\overline{M}_{(2k,l)}$ is nontrivial when $2k + l > 4$ and $l > 0$ which simply means that the moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ is not orientable in this case. □

6.1.1. Examples

In Example 4.5.3, we obtained that $\mathbb{R}\overline{M}_{(0,5)}, \mathbb{R}\overline{M}_{(2,3)}$, and $\mathbb{R}\overline{M}_{(4,1)}$ are respectively, a torus with three points blown up, a sphere with three points blown up, and a sphere with one point blown up. The coverings $\mathbb{R}\widetilde{M}_{(0,5)}, \mathbb{R}\widetilde{M}_{(2,3)}$ and $\mathbb{R}\widetilde{M}_{(4,1)}$ are obtained by taking the two copies of the corresponding moduli space of real curves and replacing the blown up loci by annuli. Therefore, $\mathbb{R}\widetilde{M}_{(0,5)}, \mathbb{R}\widetilde{M}_{(2,3)}$ and $\mathbb{R}\widetilde{M}_{(4,1)}$ are surfaces of genus 4, genus 2 and genus 0, respectively (see Figure 7 which illustrates the case $(k, l) = (0, 5)$).

6.2. Combinatorial types of strata of $\mathbb{R}\widetilde{M}_{(2k,l)}$

While constructing $\mathbb{R}\widetilde{M}_{(2k,l)}$, the closure of the each codimension one strata are glued in a consistent way. This identification of codimension strata gives an equivalence relation among the o-planar trees when $l \neq 0$.

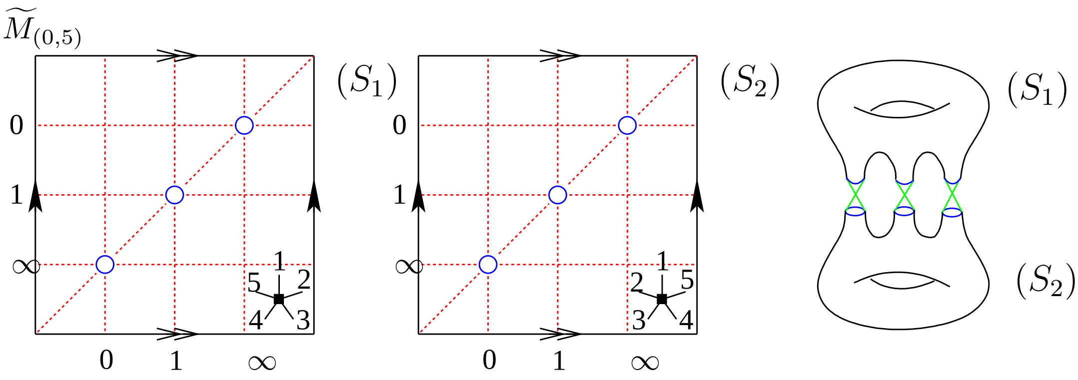

FIGURE 7. Stratification of $\mathbb{R}\widetilde{M}_{(0,5)}$

We define the notion of *R-equivalence* on the set of such o-planar trees by treating different cases separately. Let $(\gamma_1, o_1), (\gamma_2, o_2)$ be o-planar trees.

(1) If $|V^{\mathbb{R}}_{\gamma_i}| = 1$, then we say that they are R-equivalent whenever γ_1, γ_2 are isomorphic (i.e., $\gamma_1 \approx \gamma_2$) and the o-planar structures are the same.

(2) If γ_i have an edge corresponding to real node (i.e. $E^{\mathbb{R}}_{\gamma_i} = \{e\}$ and $V^{\mathbb{R}}_{\gamma_i} = \partial_\gamma(e) = \{v^e, v_e\}$), we first obtain $(\gamma_i(e), o_i(e))$ by contracting conjugate pairs of edges until there will be none. We say that (γ_1, o_1) and (γ_2, o_2) are R-equivalent whenever $\gamma_1 \approx \gamma_2$ and
- $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 1 \mod 2$,
- $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 0 \mod 2$,
- $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 1 \mod 2$,
- $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 0 \mod 2$,
- $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ and $|F^{\mathbb{R}}_\gamma(v^e)| \neq 1$,
- $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ and $|F^{\mathbb{R}}_\gamma(v^e)| = 1$,

(3) Otherwise, if γ_i have more than one invariant edge (i.e. $|E_{\gamma_i}|^{\mathbb{R}} > 1$), we say that (γ_1, o_1), (γ_2, o_2) are R-equivalent whenever $\gamma_1 \approx \gamma_2$ and there exists an edge $e \in E^{\mathbb{R}}_{\gamma_i}$ such that the o-planar trees $(\gamma_i(e), o_i(e))$, which are obtained by contracting all edges but e, are R-equivalent in the sense of the Case (2).

We call the maximal set of pairwise R-equivalent o-planar trees by *R-equivalence classes* of o-planar trees.

Theorem 6.3. *A stratification of the orientation double cover* $\mathbb{R}\widetilde{M}_{(2k,l)}$ *is given by*

$$\mathbb{R}\widetilde{M}_{(2k,l)} = \bigsqcup_{\substack{\text{R-equivalance classes} \\ \text{of o-planar } (\gamma,o)}} C_{(\gamma,o)}.$$

6.3. Some other double coverings of $\mathbb{R}\overline{M}_{(2k,l)}$

In [11], Kapranov constructed a different double covering $\widehat{\mathbb{R}M}_{(0,l)}$ of $\mathbb{R}\overline{M}_{(0,l)}$ having no boundaries. He has applied the following recipe to obtain the double covering: Let $\overline{C}_{(0,l)}$ be the disjoint union of closed strata $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ as above. Let $(\gamma, \delta(o_i)), i = 1, 2$ be two-vertex o-planar trees representing the same u-planar tree (γ, u), and let (τ, o_i) be the one-vertex trees obtained by contracting the edges of $(\gamma, \delta(o_i))$. The strata $\overline{C}_{(\gamma,\delta(o_i))}, i = 1, 2$ are glued if $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at vertex $v^e \neq \partial_\gamma(n)$. We obtain first Stiefel-Whitney class of $\widehat{\mathbb{R}M}_{(0,l)}$ by using the same arguments in Theorem 5.7.

Proposition 6.4. *The Poincaré dual of the first Stiefel-Whitney class of* $\widehat{\mathbb{R}M}_{(0,l)}$ *is*

$$[\widehat{w_1}] = \frac{1}{2} \sum_{(\tau,o)} \sum_{(\gamma,\delta(o)):|v^e|=0 \mod 2} \left[\overline{C}_{(\gamma,\delta(o))}\right] \mod 2.$$

It is well-known that these spaces are not orientable when $l \geq 5$.

6.3.1. A double covering from open-closed string theory

In [5, 17], a different 'orientation double covering' is considered. It can be given as the disjoint union $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ where $F_\tau^+ = \{1, \cdots, k\}$, and $F_\tau^{\mathbb{R}}$ carries all possible oriented cyclic ordering. It is a disjoint union of manifolds with corners. The covering map $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)} \to \mathbb{R}\overline{M}_{(2k,l)}$ is two-to-one only over a subset of the open space $\mathbb{R}M_{(2k,l)}$. It only covers the subset $\bigsqcup_{(\tau,u)} \overline{C}_{(\tau,u)}$ of $\mathbb{R}\overline{M}_{(2k,l)}$ where u-planar trees (τ, u) have the partition $\{\{1, \cdots, k\}, \{k+1, \cdots, 2k\}\}$ of $F_\tau \setminus F_\tau^{\mathbb{R}}$. Moreover, the covering map is not two-to-one over the strata with codimension higher than zero.

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Moduli spaces of rational tropical curves

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Abstract. This note is devoted to the definition of moduli spaces of rational tropical curves with n marked points. We show that this space has a structure of a smooth tropical variety of dimension $n-3$. We define the Deligne-Mumford compactification of this space and tropical ψ-class divisors.

This paper gives a detailed description of the moduli space of tropical rational curves mentioned in [4]. The survey [4] was prepared under rather sharp time and volume constraints. As a result the coordinate presentation of this moduli space from [4] contains a mistake (it was over-simplified). In this paper we'll correct the mistake and give a detailed description on $\overline{\mathcal{M}}_{0,5}$ as our main example.

1. Introduction: smooth tropical varieties

In this section we follow the definitions of [5] and [4].

The underlying algebra of tropical geometry is given by the semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ of tropical numbers. The tropical arithmetic operations are "$a+b$" $= \max\{a,b\}$ and "ab" $= a+b$. The quotation marks are used to distinguish between the tropical and classical operations. With respect to addition $\mathbb{T}$ is a commutative semigroup with zero "$0_{\mathbb{T}}$" $= -\infty$. With respect to multiplication $\mathbb{T}^\times = \mathbb{T} \smallsetminus \{0_{\mathbb{T}}\} \approx \mathbb{R}$ is an honest commutative group with the unit "$1_{\mathbb{T}}$" $= 0$. Furthermore, the addition and multiplication satisfy the distribution law "$a(b+c)$" $=$ "$ab+ac$", $a,b,c \in \mathbb{T}$. These operations may be viewed as a result of the so-called *dequantization* of the classical arithmetic operations that underlies the *patchworking* construction, see [3] and [8].

These tropical operations allow one to define tropical Laurent polynomials. Namely, a tropical Laurent polynomial is a function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(x) = \text{“}\sum_j a_j x^j\text{”} = \max_j (a_j + jx),$$

where jx denotes the scalar product, $x \in (\mathbb{T}^\times)^n \approx \mathbb{R}^n$, $j \in \mathbb{Z}^n$ and only finitely many coefficients $a_j \in \mathbb{T}$ are non-zero (i.e. not $-\infty$).

Affine-linear functions with integer slopes (for brevity we call them simply *affine functions*) form an important subcollection of all Laurent polynomials. Namely, these are such functions $f : \mathbb{R}^n \to \mathbb{R}$ that both f and "$\frac{1_T}{f}$" $= -f$ are tropical Laurent polynomials.

We equip $\mathbb{T}^n \approx [-\infty, \infty)^n$ with the Euclidean topology. Let $U \subset \mathbb{T}^n$ be an open set.

Definition 1.1. A continuous function $f : U \to \mathbb{T}$ is called *regular* if its restriction to $U \cap \mathbb{R}^n$ coincides with a restriction of some tropical Laurent polynomial to $U \cap \mathbb{R}^n$.

We denote the sheaf of regular functions on $\mathbb{T}^n$ with $\mathcal{O}$ (or sometimes $\mathcal{O}_{\mathbb{T}^n}$ to avoid confusion). Any subset $X \subset \mathbb{T}^n$ gets an induced regular sheaf $\mathcal{O}_X$ by restriction. For our purposes we restrict our attention only to the case when X is a polyhedral complex, i.e. when X is the closure of a union of convex polyhedra (possibly unbounded) in $\mathbb{R}^n$ such that the intersection of any number of such polyhedra is their common face. We say that X is an k-dimensional polyhedral complex if it is obtained from a union of k-dimensional polyhedra. These polyhedra are called the *facets* of X.

Let $V \subset X$ be an open set and $f \in \mathcal{O}_X(V)$ be a regular function in V. A point $x \in V$ is called a "zero point" of f if the restriction of "$\frac{1_{\mathbb{T}}}{f}$" $= -f$ to $W \subset V$ is not regular for any open neighborhood $W \ni x$. Note that it may happen that x is a "zero point" for $\phi : U \to \mathbb{T}$, but not for $\phi|_{X \cap U}$. It is easy to see that if X is a k-dimensional polyhedral complex then the "zero locus" Z_f of f is a $(k-1)$-dimensional polyhedral subcomplex.

To each facet of Z_f we may associate a natural number, called its *weight* (or degree). To do this we choose a "zero point" x inside such a facet. We say that x is a "simple zero" for f if for any local decomposition into a sum (i.e. the tropical product) of regular function $f =$ "gh" $= g+h$ on V near x we have either g or h affine (i.e. without a "zero"). We say that the weight is l if f can be locally decomposed into a tropical product of l functions with a simple zero at x.

A regular function f allows us to make the following modification on its domain $V \subset X \subset \mathbb{T}^n$. Consider the graph

$$\Gamma_f \subset V \times \mathbb{T} \subset \mathbb{T}^{n+1}.$$

It is easy to see that the "zero locus"

$$\bar{\Gamma}_f \subset V \times \mathbb{T}$$

of the (regular) function "$y + f(x)$" (defined on $V \times \mathbb{T}$), where x is the coordinate on V and y is the coordinate on $\mathbb{T}$, coincides with the union of Γ_f and the undergraph

$$U\Gamma_{f,Z} = \{(x, y) \in V \times \mathbb{T} \mid x \in Z_f, y \leq f(x)\}.$$

Furthermore, the weight of a facet $F \subset \bar{\Gamma}_f$ is 1 if $F \in \Gamma_f$ (recall that as V is an unweighted polyhedral complex all the weights of its facets are equal to one) and is the weight of the corresponding facet of Z_f if $F \in U\Gamma_{f,Z}$. We view $\bar{\Gamma}_f$ as a "tropical closure" of the set-theoretical graph Γ_f. Note that we have a map $\bar{\Gamma}_f \to V$. We set $\tilde{V} = \bar{\Gamma}_f$ to be the result of the *tropical modification* $\mu_f : \tilde{V} \to V$ along the regular function f. The locus Z_f is called the *center* of tropical modification.

The weights of the facets of $\tilde{V}$ supplies us with some inconvenience as they should be incorporated in the definition of the regular sheaf $\mathcal{O}_{\tilde{V}}$ on $\tilde{V}$. Namely, the affine functions defined by $\mathcal{O}_{\tilde{V}}$ on a facet of weight w should contain the group of functions that come as restrictions to this facet of the affine functions on $\mathbb{T}^{n+1}$ as a subgroup of index w.

Sometimes one can get rid of the weights of $\tilde{V}$ by a reparameterization with the help of a map $\bar{V} \to \tilde{V}$ that is given by locally linear maps in the corresponding charts. Indeed, the restriction of $\mu_g : \bar{V} \to \tilde{V}$ to a facet is locally given by a linear function between two k-dimensional affine-linear spaces defined over $\mathbb{Z}$. If its determinant equals to w then the push-forward of $\mathcal{O}_{\bar{V}}$ supplies an extension of $\mathcal{O}_{\tilde{V}}$ required by the weights. Note however that if $w > 1$ then the converse map is not defined over $\mathbb{Z}$ and thus is not given by elements of $\mathcal{O}_{\tilde{V}}$.

Tropical modifications give the basic equivalence relation in Tropical Geometry. It can be shown that if we start from $\mathbb{T}^k$ and do a number of tropical modifications on it then the result is a k-dimensional polyhedral complex $Y \subset \mathbb{T}^n$ that satisfies to the following *balancing property* (cf. Property 3.3 in [4] where balancing is restated in an equivalent way).

Property 1.2. Let $E \subset Y \cap \mathbb{R}^N$ be a $(k-1)$-dimensional face and $F_1, \dots, F_l$ be the facets of Y adjacent to F whose weights are $w_1, \dots, w_l$. Let $L \subset \mathbb{R}^N$ be a $(N-k)$-dimensional affine-linear space with an integer slope and such that it intersects E. For a generic (real) vector $v \in \mathbb{R}^N$ the intersection $F_j \cap (L+v)$ is either empty or a single point. Let $\Lambda_{F_j} \subset \mathbb{Z}^N$ be the integer vectors parallel to F_j and $\Lambda_L \subset \mathbb{Z}^N$ be the integer vectors parallel to L. Set λ_j to be the product of w_j and the index of the subgroup $\Lambda_{F_j} + \Lambda_L \subset \mathbb{Z}^N$. We say that $Y \subset \mathbb{T}^n$ is *balanced* if for any choice of E, L and a small generic v the sum

$$\iota_L = \sum_{j \ | \ F_j \cap (L+v) \neq \emptyset} \lambda_j$$

is independent of v. We say that Y is *simply balanced* if in addition for every j we can find L and v so that $F_j \cap (L+v) \neq \emptyset$, $\iota_L = 1$ and for every small v there exists an affine hyperplane $H_v \subset L$ such that the intersection $Y \cap (L+v)$ sits entirely on one side of $H_v + v$ in $L + v$ while the intersection $Y \cap (H_v + v)$ is a point.

Definition 1.3 (cf. [5],[4])**.** A topological space X enhanced with a sheaf of tropical functions $\mathcal{O}_X$ is called a (smooth) tropical variety of dimension k if for every $x \in X$ there exist an open set $U \ni x$ and an open set V in a simply balanced polyhedral complex $Y \subset \mathbb{T}^N$ such that the restrictions $\mathcal{O}_X|_U$ and $\mathcal{O}_Y|_V$ are isomorphic.

Tropical varieties are considered up to the equivalence generated by tropical modifications. It can be shown that a smooth tropical variety of dimension k can be locally obtained from $\mathbb{T}^k$ by a sequence of tropical modifications centered at smooth tropical varieties of dimension $(k-1)$. This follows from the following proposition.

Proposition 1.4. *Any k-dimensional simply balanced polyhedral complex $X \subset \mathbb{R}^n$ can be obtained from $\mathbb{T}^k$ by a sequence of consecutive tropical modifications whose centers are simply balanced $(k-1)$-dimensional polyhedral complexes.*

Proof. We prove this proposition inductively by n. Without the loss of genericity we may assume that X is a fan, i.e. each convex polyhedron of X is a cone centered at the origin.

The base of the induction, when $n = k$, is trivial. When $n > k$ let us take an $(n-k)$-dimensional affine-linear subspace $L \subset \mathbb{R}^n$ given by Property 1.2. Choose a linear projection

$$\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1}$$

defined over $\mathbb{Z}$ and such that $\ker(\lambda)$ is a line contained in L.

The image $\lambda(X) \subset \mathbb{R}^{n-1}$ is a k-dimensional polyhedral complex since L is transversal to some facets of X. We claim that

$$\lambda|_X : X \to \lambda(X)$$

is a tropical modification once we identify $\mathbb{R}^n$ and $\mathbb{R}^{n-1} \times \mathbb{R}$. The center of this modification is the locus

$$Z_f = \{x \in \mathbb{R}^{n-1} \mid \dim(\lambda^{-1}(x) \cap X) > 0\}.$$

Here we use the dimension in the usual topological sense. Note that the $(k-1)$-dimensional complex $Z_f \subset \mathbb{R}^{n-1}$ is simply balanced, existence of the needed $(n-k)$-dimensional affine-linear spaces follows from the fact that $X \subset \mathbb{R}^n$ is simply balanced.

To justify our claim we note that near any point $x \in Z_f$ the subcomplex $Y \subset X$ obtained as the (Euclidean topology) closure of $X \smallsetminus \lambda^{-1}(Z_f)$ is a (set-theoretical) graph of a convex function. This, once again, follows from the fact that $X \subset \mathbb{R}^n$ is simply balanced, this time applied to the points in the facets on $X \smallsetminus Y$. Thus it gives a regular tropical function f and it remains only to show that the the weight of any facet of $E \subset Z_f$ is 1. But this follows, in turn, from the balancing condition at $\lambda^{-1}(E) \cap Y$. □

2. Tropical curves and their moduli spaces

The definition of tropical variety is especially easy in dimension 1. Tropical modifications take a graph into a graph (with arbitrary valence of its vertices) and the tropical structure carried by the sheaf $\mathcal{O}_X$ amounts to a complete metric on the complement of the set of 1-valent vertices of the graph X (cf. [5], [6], [1]). Thus, each 1-valent vertex of a tropical curve X is adjacent to an edge of infinite length.

A tropical modification allows one to contract such an edge or to attach it at any point of X other than a 1-valent vertex. If we have a finite collection of marked points on X then by passing to an equivalent model if needed we may assume that the set of marked points coincides with the set of 1-valent vertices. (Of course, if X is a tree then we have to have at least two marked points to make such assumption.)

The *genus* of a tropical curve X is $\dim H_1(X)$. Let $\mathcal{M}_{g,n}$ be the set of all tropical curves X of genus g with n distinct marked points. Fixing a combinatorial type of a graph Γ with n marked leaves defines a subset $U_\Gamma \subset \mathcal{M}_{g,n}$ consisting of marked tropical curves with this combinatorics. A length of any non-leaf edge of Γ defines a real-valued function on U_Γ. Such functions are called *edge-length functions*. To avoid difficulties caused by self-automorphisms of X from now on we restrict our attention to the case $g = 0$.

Definition 2.1. The *combinatorial type* of a tropical curve X is its equivalence class up to homeomorphisms respecting the markings.

Combinatorial types partition the set $\mathcal{M}_{0,n}$ into disjoint subsets. The edge-length functions define the structure of the polyhedral cone $\mathbb{R}^M_{\geq 0}$ in each of those subsets (as the lengths have to be positive). The number M here is the number of the bounded (non-leaf) edges in X. By the Euler characteristic reasoning it is equal to $n-3$ if X is (1- and) 3-valent, it is smaller if X has vertices of higher valence.

Furthermore, any face of the polyhedral cone $\mathbb{R}^M_{\geq 0}$ coincides with the cone corresponding to another combinatorial type, the one where we contract some of the edges of X to points. This gives the adjacency (fan-like) structure on $\mathcal{M}_{0,n}$, so $\mathcal{M}_{0,n}$ is a (non-compact) polyhedral complex. In particular, it is a topological space.

Theorem 1. *The set $\mathcal{M}_{0,n}$ for $n \geq 3$ admits the structure of an $(n-3)$-dimensional tropical variety such that the edge-length functions are regular within each combinatorial type. Furthermore, the space $\mathcal{M}_{0,n}$ can be tropically embedded in $\mathbb{R}^N$ for some N (i.e. $\mathcal{M}_{0,n}$ can be presented as a simply balanced complex).*

Proof. This theorem is trivial for $n = 3$ as $\mathcal{M}_{0,3}$ is a point. Otherwise, any two disjoint ordered pairs of marked points can be used to define a global regular function on $\mathcal{M}_{0,n}$ with values in $\mathbb{R} = \mathbb{T}^\times$. Namely, each such ordered pair defines the oriented path on the tropical curve X connecting the corresponding marked points. These paths can be embedded.

Since the two pairs of marked points are disjoint, the intersection of the two corresponding paths has to have finite length. We take this length with the positive sign if the orientations agree and with the negative sign otherwise. This defines a function on $\mathcal{M}_{0,n}$. We call such functions the *double ratio* functions.

Take all possible disjoint pairs of marked points and use them as coordinates for our embedding

$$\iota : \mathcal{M}_{0,n} \to \mathbb{R}^N,$$

where N is the number of all possible decompositions of n into two disjoint pairs. The theorem now follows from the following two lemmas. □

Note that, strictly speaking, each coordinate in $\mathbb{R}^N$ depends not only on the choice of two disjoint pairs of marked points but also on the order of points in each pair. However, changing the order in one of the pairs only reverses the sign of the double ratio. Taking an extra coordinate for such a change of order would be redundant. Indeed, for any balanced complex $Y \subset \mathbb{R}^N$ and any affine-linear function $\lambda : \mathbb{R}^N \to \mathbb{R}$ with an integer slope the graph of λ is a balanced complex in $\mathbb{R}^{N+1}$ isomorphic to the initial complex Y.

Lemma 2.2. *The map ι is a topological embedding.*

Proof. First, let us prove that ι is an embedding. The combinatorial type of X is determined by the set of the coordinates that do not vanish on X. Indeed, any non-leaf edge E of the tree X separates the leaves (i.e. the set of markings) into two classes corresponding to the components of $X \setminus E$. Let us take a coordinate in $\mathbb{R}^n$ that corresponds to four marking points (union of the two disjoint pairs) such that two of these points belong to one class and two to the other class. We call such a coordinate an *E-compatible coordinate*. Note that an E-compatible coordinate vanishes on X if and only if the pairs of markings defined by the coordinate agree with the pairs defined by the classes.

This observation suffices to reconstruct the combinatorial type of X. Furthermore, the length of E equals to the minimal non-zero absolute value of the E-compatible coordinates. This implies that ι is an embedding. $\square$

Lemma 2.3. *The image $\iota(\mathcal{M}_{0,n})$ is a simply balanced complex in $\mathbb{R}^N$.*

Proof. This is a condition on codimension 1 faces of $\mathcal{M}_{0,n}$. First we shall check it for the case $n = 4$. There are three ways to split the four marking points into two disjoint pairs. Accordingly, there are three combinatorial types of 3-valent trees with three marked leaves. Thus our space $\mathcal{M}_{0,4}$ is homeomorphic to the *tripod*, or the "interior" of the letter Y, see Figure 1. Each ray of this tripod correspond to a combinatorial type of a 3-valent tree with 4 leaves while the vertex correspond to the 4-valent tree.

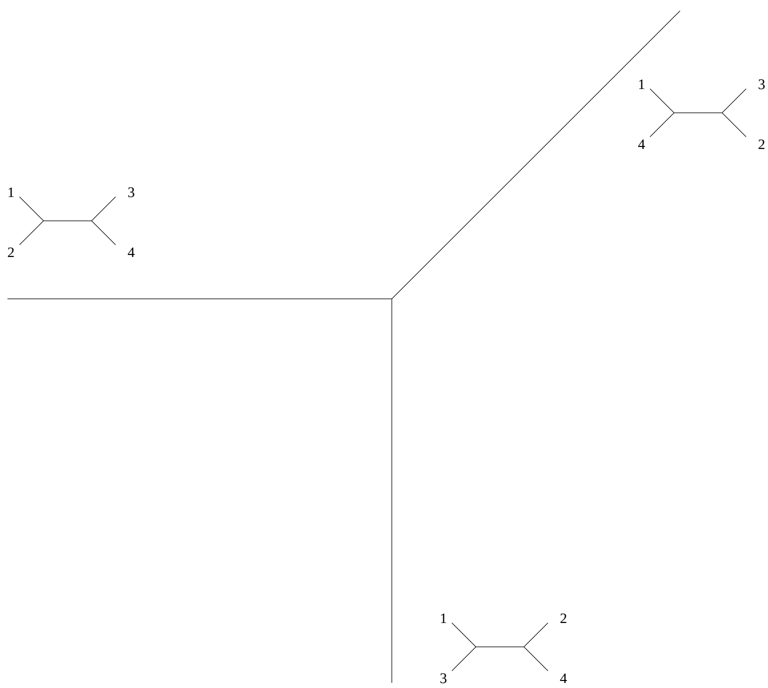

Figure 1. The tropical moduli space $\mathcal{M}_{0,4}$ and its points on the corresponding edges.

Up to the sign we have the total of three double ratios for $n = 4$. Let us e.g. take those defined by the following ordered pairs: $\{(12),(34)\}$, $\{(13),(24)\}$ and $\{(14),(23)\}$ Each is vanishing on the corresponding ray of the tripod. Let us parameterize each ray of the tripod by its only edge-length $t \geq 0$ and compute the corresponding map to $\mathbb{R}^3$.

We have the following embeddings on the three rays

$$t \mapsto (0,t,t),\ t \mapsto (t,0,-t),\ t \mapsto (-t,-t,0).$$

The sum of the primitive integer vectors parallel to the resulting directions is 0 and thus $\iota(\mathcal{M}_{0,4})$ is balanced.

In the case $n > 4$ the codimension 1 faces of $\mathcal{M}_{0,n}$ correspond to the combinatorial types of X with a single 4-valent vertex. Near a point inside of such face F the space $\mathcal{M}_{0,n}$ looks like the product of $\mathcal{M}_{0,4}$ and $\mathbb{R}^{n-4}$. The factor $\mathbb{R}^{n-4}$ comes from the edge-lengths on F (its combinatorial type has $n-4$ bounded edges) while the factor $\mathcal{M}_{0,4}$ comes from perturbations of the 4-valent vertex (which result in a new bounded edge in one of the three possible combinatorial types of the result).

We have a well-defined map from the union U of the F-adjacent facets to F by contracting the new edge to a point. Note that the edge-length functions exhibit F as the positive quadrant in $\mathbb{R}^{n-4}$. Furthermore, in the combinatorial type of F we may choose 4 leaves such that contracting all other leaves will take place outside of the 4-valent vertex (see Figure 2). This contraction defines a map $U \to \mathcal{M}_{0,4}$.

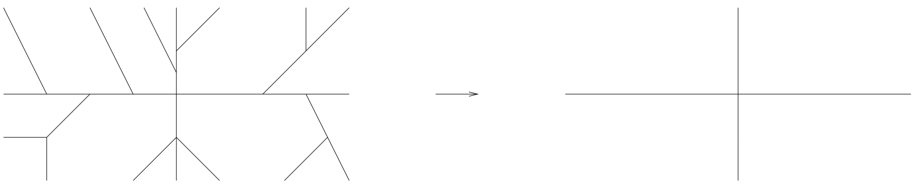

Figure 2. One of the possible contractions of a tree with a 4-valent vertex to the tree corresponding to the origin $O \in \mathcal{M}_{0,4}$.

The lemma now follows from the observation that the resulting decomposition into $\mathcal{M}_{0,4} \times \mathbb{R}^{n-4}$ agrees with the double ratio functions. Indeed, note that the complement of the 4-valent vertex for a curve in the combinatorial type F is composed of four components. If the double ratio is such that its four markings are in one-to-one correspondence with these components then at U it coincides with sum of the pull-back of the corresponding double ratio in F with the pull-back of the corresponding double ratio in $\mathcal{M}_{0,4}$. If one of the four components is lacking a marking from the double ratio ρ then $\rho|_U$ coincides with the corresponding pull-back from F. □

Remark 2.4. The functions Z_{x_i,x_j} from [4] do not define regular functions on $\mathcal{M}_{0,n}$, contrary to what is written in [4]. These functions were a result of an erroneous simplification of the double ratio functions. But these functions cannot be regular as they are always positive and Proposition 5.12 of [4] is not correct. Even the projectivization of the embedding is not a balanced complex already for $\mathcal{M}_{0,5}$. One should use the (non-simplified) double ratios instead.

Clearly, the space $\mathcal{M}_{0,n}$ is non compact. However it is easy to compactify it by allowing the lengths of bounded edges to assume infinite values. Let $\overline{\mathcal{M}}_{0,n}$ be the space of connected trees with n (marked) leaves such that each edge of this tree is assigned a length $0 < l \leq +\infty$ so that each leaf has length necessarily equal to $+\infty$.

Corollary 2.5. *The space $\overline{\mathcal{M}}_{0,n}$ is a smooth compact tropical variety.*

To verify that $\overline{\mathcal{M}}_{0,n}$ is smooth near a point x at the boundary

$$\partial\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n} \smallsetminus \mathcal{M}_{0,n}$$

we need to examine those double ratios that are equal to $\pm\infty$ at x. There we use only those signs that result in $-\infty$ do that the map takes values in $\mathbb{T}^N$.

Remark 2.6. Note that the compactification $\overline{\mathcal{M}}_{0,n} \supset \mathcal{M}_{0,n}$ corresponds to the Deligne-Mumford compactification in the complex case as under the 1-parametric family collapse of a Riemannian surface to a tropical curve the tropical length of an edge corresponds to the rate of growth of the complex modulus of the holomorphic annulus collapsing to that edge.

Furthermore, similarly to the complex story the infinite edges decompose a tropical curve into components (where the non-leaf edges are finite). Any tropical map from an infinite edge which is bounded would have to be constant and thus the image would have to split as a union of several tropical curves in the target. Such decompositions were used by Gathmann and Markwig in their deduction of the tropical WDVV equation in $\mathbb{R}^2$, see [1].

3. Tropical ψ-classes

Note that we do have the forgetting maps

$$\mathrm{ft}_j : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$$

for $j = 1, \dots, n+1$ by contracting the leaf with the j-marking. This map is sometimes called the *universal curve.* Each marking $k \neq j$ defines a section σ_k of ft_j. The conormal bundle to σ_k defines the ψ_k-class in complex geometry (to avoid ambiguity we take $j = n+1$). This notion can be adapted to our tropical setup.

Recall that so far our choice of tropical models in their equivalence class was such that the leaves of the tropical curves were in 1-1 correspondence with the markings. For this choice we have the images $\sigma_k(\overline{\mathcal{M}}_{0,n})$ contained in the boundary part of $\overline{\mathcal{M}}_{0,n+1}$. This presentation is compatible with the point of view when we think about line bundles in tropical geometry to be given by $H^1(X, \mathcal{O}^\times)$. Here X is the base of the bundle and $\mathcal{O}^\times$ is the sheaf of "non-vanishing" tropical regular functions. Such functions are given in the charts to $\mathbb{R}^N$ by affine-linear functions with integer slopes, see [6]. (Recall that $\mathbb{T}^\times = \mathbb{R}$ is an honest group with respect to tropical multiplication, i.e. the classical addition.)

However, the following alternative construction allows one to obtain the ψ-classes more geometrically (as we'll illustrate in an example in the next section). This approach is based on contracting the leaves marked by number k.

The canonical class of a tropical curve is supported at its vertices, namely we take each vertex with the multiplicity equal to its valence minus 2, cf. [6]. Furthermore, the cotangent bundle near a 3-valent vertex point can be viewed as a neighborhood of the origin for the line given by the tropical polynomial "$x + y + 1_{\mathbb{T}}$" in $\mathbb{R}^2$, so the $+1$ self-intersection of the line gives the required multiplicity for the canonical class at any 3-valent vertex. Thus we can use the intersections with the corresponding codimension 1 faces in $\mathcal{M}_{0,n}$ to define the ψ-classes there. In other words, tropical ψ-classes will be supported on the $(n-4)$-dimensional faces in $\mathcal{M}_{0,n}$.

Namely, for a ψ_k-class we have to collect those codimension 1 faces in $\mathcal{M}_{0,n}$ whose only 4-valent vertex is adjacent to the leaf marked by k. After a contraction of this leaf we get a 3-valent vertex, thus the multiplicity of every face in a ψ-divisor is 1. We arrive at the following definition.

Definition 3.1. The tropical ψ_k-divisor $\Psi_k \subset \mathcal{M}_{0,n}$ is the union of those $(n-4)$-dimensional faces that correspond to tropical curves with a 4-valent vertex adjacent to the leaf marked by k, $k = 1, \dots, n$. Each such face is taken with the multiplicity 1.

Proposition 3.2. *The subcomplex Ψ_k is a divisor, i.e. satisfies the balancing condition.*

Proof. Recall that the balancing condition is a condition at $(n-5)$-dimensional faces. In $\mathcal{M}_{0,n}$ there are two types of such faces, one corresponding to tropical curves with two 4-valent vertices and one corresponding to a tropical curve with a 5-valent vertex.

Near the faces of the first type the moduli space $\mathcal{M}_{0,n}$ is locally a product of two copies of $\mathcal{M}_{0,4}$ and $\mathbb{R}^{n-5}$. The Ψ-divisor is a product of $\mathbb{R}^{n-5}$, one copy of $\mathcal{M}_{0,4}$ and the central (3-valent) point in the other copy of $\mathcal{M}_{0,4}$ (this is the point corresponding to the 4-valent vertex adjacent to the leaf marked by k). Thus the balancing condition holds trivially in this case.

Near the faces of the second type the moduli space $\mathcal{M}_{0,n}$ is locally a product of $\mathcal{M}_{0,5}$ and $\mathbb{R}^{n-5}$. As in the proof of Theorem 1 each double ratio decomposes to the sum of the corresponding double ration in $\mathcal{M}_{0,5}$ (perhaps trivial if two of the markings for the double ratio correspond to the same edge adjacent to the 5-valent vertex) and an affine-linear function in $\mathbb{R}^{n-5}$. Thus it suffices to check only the balancing condition for the Ψ-divisors in $\mathcal{M}_{0,5}$. This example is considered in details in the next section. The balancing condition there follows from Proposition 4.1. □

Conjecturally, the tropical Ψ-divisors are limits of some natural representatives of the divisors for the complex ψ-classes under the collapse of the complex moduli space onto the corresponding tropical moduli space $\mathcal{M}_{0,n}$. Note that our choice for the tropical Ψ-divisor is not contained in the boundary $\partial\overline{\mathcal{M}}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$ (cf. the calculus of the complex boundary classes in [2]), but comes as a closure of a divisor in $\mathcal{M}_{0,n}$.

4. The space $\overline{\mathcal{M}}_{0,5}$

We have already described the moduli space $\mathcal{M}_{0,4}$ as the tripod of Figure 1. It has only one 0-dimensional face $O \in \mathcal{M}_{0,4}$. This point (considered as a divisor) coincides with the divisors $\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4$. The description of $\mathcal{M}_{0,5}$ is somewhat more interesting.

There are 15 combinatorial types of 3-valent trees with 5 marked leaves. If we forget about the markings there is only one homeomorphism class for such a curve (see Figure 3). To get the number of non-isomorphic markings we take the number all possible reordering of vertices (equal to $5! = 120$) and divide by $2^3 = 8$ as there is an 8-fold symmetry of reordering. Indeed there is one symmetry interchanging the left two leaves, one interchanging the right two leaves and the central symmetry around the central leave of the 3-valent tree on top of Figure 3.

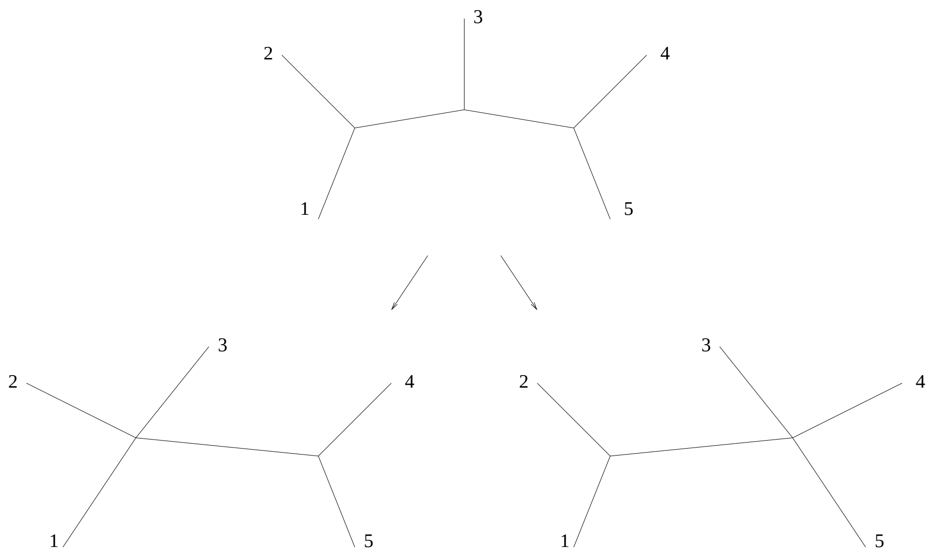

FIGURE 3. Adjunction of combinatorial types corresponding to the quadrant connecting the rays (45) and (12).

Thus the space $\mathcal{M}_{0,5}$ is a union of 15 quadrants $\mathbb{R}^2_{\geq 0}$. These quadrants are attached along the rays which correspond to the combinatorial types of curves with one 4-valent vertex. Such curves also have one 3-valent vertex which is adjacent to two leaves and the only bounded edge of the curve, see the bottom of Figure 3. Such combinatorial types are determined by the markings of the two leaves emanating from the 3-valent vertex. Thus we have a total of $\binom{5}{2} = 10$ of such rays.

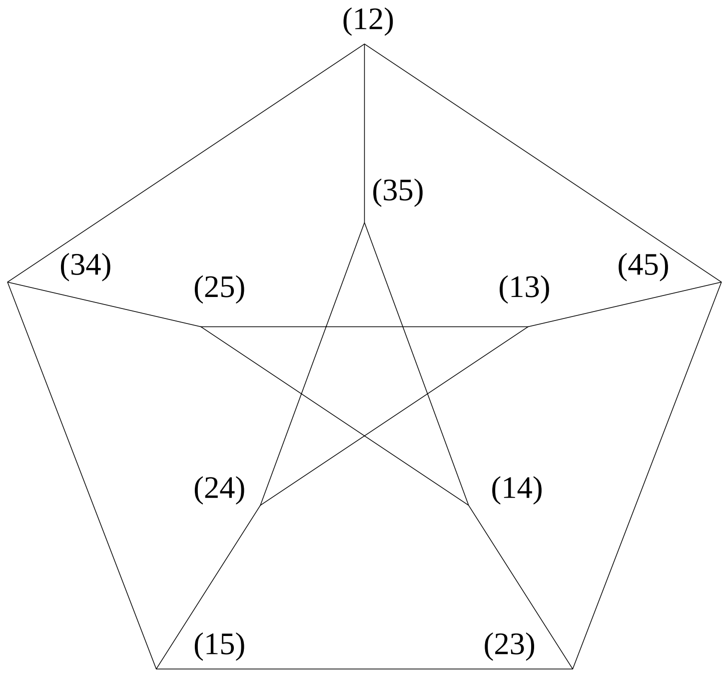

FIGURE 4. The link of the origin in $\mathcal{M}_{0,5}$.

The two boundary edges of the quadrant correspond to contractions of the bounded edges of the combinatorial type as shown on Figure 3. The global picture of adjacency of quadrants and rays is shown on Figure 4 where the reader may recognize the well-known *Petersen graph*, cf. the related tropical Grassmannian picture in [7]. Vertices of this graph correspond to the rays of $\mathcal{M}_{0,5}$ while the edges correspond to the quadrants. Thus the whole picture may be interpreted as the link of the only vertex $O \in \mathcal{M}_{0,5}$ (the point O corresponds to the tree with a 5-valent vertex adjacent to all the leaves).

To locate the Ψ_k-divisor we recall that the kth leaf has to be adjacent to a 4-valent vertex if it appears in Ψ_k. This means that Ψ_k consists of 6 rays that are marked by pairs not containing k.

Proposition 4.1. *The subcomplex $\Psi_k \subset \mathcal{M}_{0,5}$ is a divisor.*

Proof. Since the whole $\mathcal{M}_{0,5}$ is S_5-symmetric it suffices to check the balancing condition only for Ψ_1. The embedding $\mathcal{M}_{0,5} \subset \mathbb{R}^N$ is given by the double ratios, so it suffices to check that for each double ratio function the sum of its gradients on the six rays of Ψ_1 vanishes.

If the double ratio is determined by two pairs disjoint from the marking 1, e.g. by $\{(23),(45)\}$ then its restriction onto the six rays of Ψ_1 is the same as its restriction to the three rays $\mathcal{M}_{0,4}$ taken twice and thus balanced. Namely its gradient is 1 on the rays (24) and (35); -1 on the rays (25) and (34); and 0 on the rays (23) and (45).

If the four markings of the double ratio contain the marking 1 then thanks to the symmetry we may assume that the double ratio is given by $\{(12),(34)\}$. It vanishes on the rays (34), (35), (45) and (25); it has gradient $+1$ on the ray (24) and the gradient -1 on the ray (23). Once again, the balancing condition holds. □

As our final example of the paper we would like to describe explicitly the universal curve

$$\mathrm{ft}_5 : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}.$$

This is presented on Figure 5. Once again, we interpret the Peterson graph as the link L

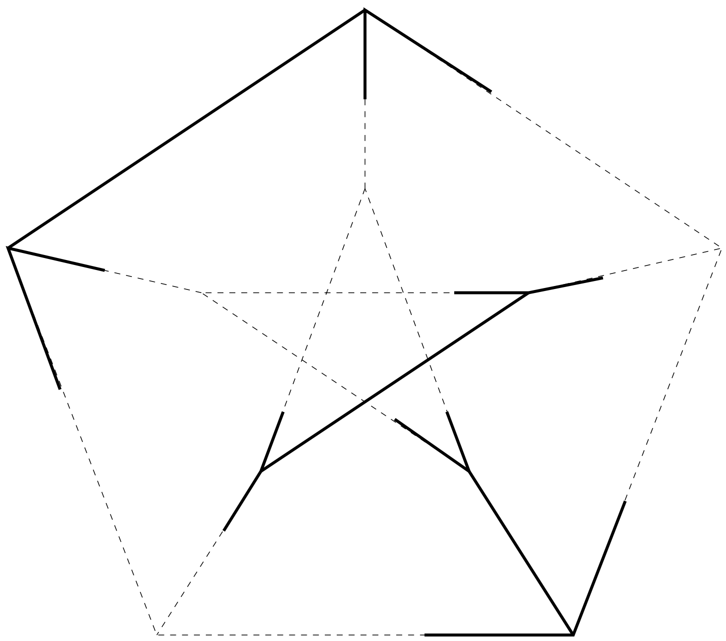

FIGURE 5. The three fibers and four sections of the universal curve $\mathrm{ft}_5 : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$.

of the vertex $O \in \mathcal{M}_{0,5}$. Similarly, the link of the origin in $\mathcal{M}_{0,4}$ consists of three points. Thus L is the union of the fibers of ft_5 (away from a neighborhood of infinity) over these three points and four copies of a neighborhood of the origin in $\mathcal{M}_{0,4}$ corresponding to the four sections σ_1, σ_2, σ_3 and σ_4 of the universal curve. Figure 5 depicts the fibers in L with solid lines and the sections with dashed lines.

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Exploded fibrations

Brett Parker

Abstract. Initiated by Gromov in [1], the study of holomorphic curves in symplectic manifolds has been a powerfull tool in symplectic topology, however the moduli space of holomorphic curves is often very difficult to find. A common technique is to study the limiting behavior of holomorphic curves in a degenerating family of complex structures which corresponds to a kind of adiabatic limit. The category of exploded fibrations is an extension of the smooth category in which some of these degenerations can be described as smooth families.

The first part of this paper is devoted to defining exploded fibrations and a slightly more specialized category of exploded $\mathbb{T}$ fibrations. In section 6 are some examples of holomorphic curves in exploded $\mathbb{T}$ fibrations, including a brief discussion of the relationship between tropical geometry and exploded $\mathbb{T}$ fibrations. In section 7, it is shown that exploded fibrations have a good intersection theory. In section 8, the perturbation theory of holomorphic curves in exploded $\mathbb{T}$ fibrations is sketched.

1. Introduction

A symplectic manifold is a manifold M^{2n} with a closed, maximally nondegenerate two form ω, called the symplectic form. Every manifold of this type has local coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ in which the symplectic form looks like $\sum dx_i \wedge dy_i$. For this reason symplectic manifolds have no local invariants, and the study of symplectic manifolds is called symplectic topology.

One of the most powerfull tools in symplectic topology is the study of holomorphic curves. Given a symplectic manifold (M, ω), there is a contractible choice of almost complex structure J on M which is tamed by ω in the sense that $\omega(v, Jv) > 0$ for any nonzero vector v. With such a choice of J, a holomorphic curve is a map

$$f : (S, j) \longrightarrow (M, J)$$

from a Riemann surface S with a complex structure j so that $df \circ j = J \circ df$. (These are sometimes called pseudoholomorphic curves because (M, J) is not a complex manifold.) The energy of a holomorphic curve is $\int_S f^*\omega$. In [1], Gromov proved that the moduli space of holomorphic curves with bounded genus and energy in a compact symplectic manifold is compact. The moduli space should also be 'smooth' in some sense, and an

Key words and phrases. holomorphic curves, symplectic topology, log structure, tropical geometry, exploded fibrations, adiabatic limit.

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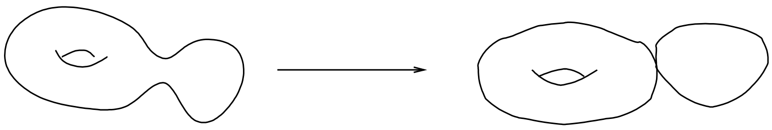
isotopy of J gives a compact cobordism of moduli spaces. This allows the definition of Gromov Witten invariants of a symplectic manifold which involve counting the number of holomorphic curves with some constraints.

The sense in which the moduli space of holomorphic curves is 'smooth' is not the usual definition. Usually a smooth family of maps corresponds to a pair of smooth maps

$$F \longleftarrow C \longrightarrow M$$

where $C \longrightarrow F$ is a surjective submersion. The individual maps in this family are the maps of the fibers of $C \longrightarrow F$ into M.

The moduli space of holomorphic curves includes families of holomorphic curves in which a bubble forms

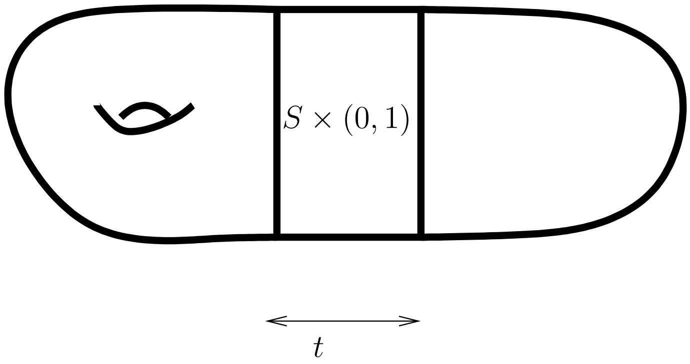


This change in the topology of the domain can not take place in a connected smooth family of maps, so if the behavior above is to be considered 'smooth', we need a new definition. This is one indication that the category of smooth manifolds might not be the natural category to describe the theory of holomorphic curves.

A second reason to try to find an extension of the smooth category with a good theory of holomorphic curves is that holomorphic curves are in general difficult to find in the non algebraic setting. Many techniques for finding holomorphic curve invariants involve degenerating the almost complex structure J in some way so that holomorphic curves become easier to find in the limit. We will now see two types of degenerations which can be considered as cutting a symplectic manifold into smaller, simpler pieces in order to compute holomorphic curve invariants. These degenerations do not have a smooth limit in the category of manifolds, however they can be considered as 'smooth' families of exploded fibrations.

One example similar to the formation of the bubble shown above is the degeneration of almost complex structure used in symplectic field theory to break a symplectic manifold apart along a hypersurface. This is described in [2].

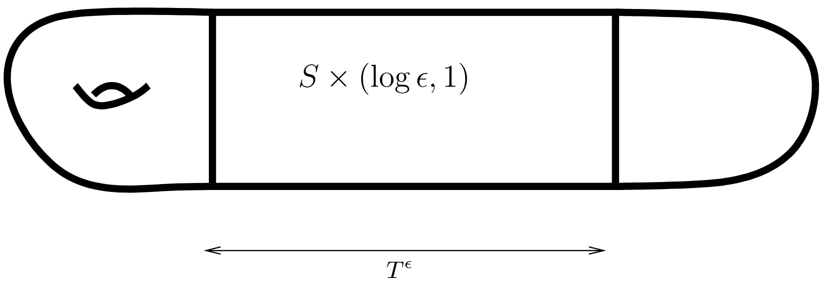
Suppose we have a hypersurface $S \subset M$ with a collar neighborhood equal to $(0,1) \times S$. Denote by t the coordinate for $(0,1)$, by X_t the Hamiltonian vector field generated by t so that $\omega(X_t, \cdot) = dt$. Suppose also that the vector field X_t on S and $\omega(\frac{\partial}{\partial t}, \cdot)$ are both independent of t. We can choose an almost complex structure J tamed by ω which is independent of t, so that $JX_t = \frac{\partial}{\partial t}$, and which preserves the kernel of $\omega(\frac{\partial}{\partial t}, \cdot)$ restricted to S.



We can then choose a degenerating family of almost complex structures J^ϵ on M as follows. Keep J^ϵ constant on the kernel of $\omega(\frac{\partial}{\partial t}, \cdot)$ restricted to S, and have

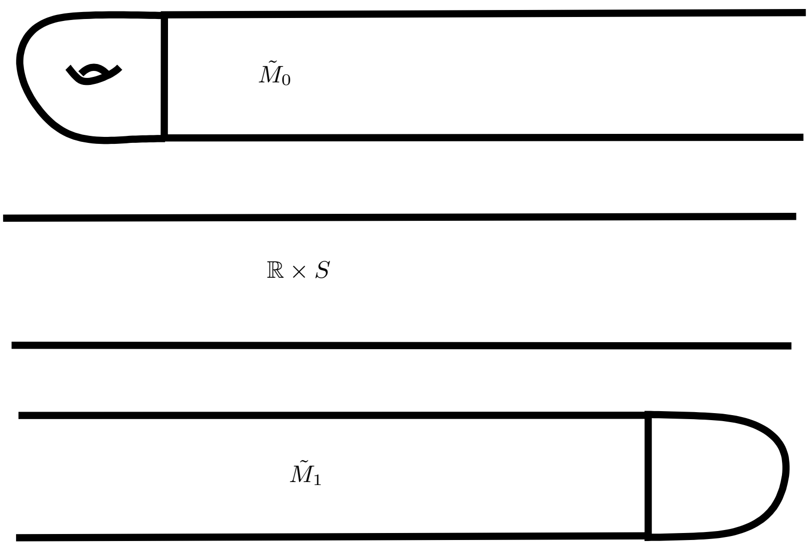
$$J^\epsilon X_t = \frac{dt}{dT^\epsilon} \frac{\partial}{\partial t}$$

where $T^\epsilon(t)$ is some family of smooth monotone increasing functions on $[0,1]$ with $\frac{dT^\epsilon}{dt} = 1$ in a neighborhood of 0 and 1 and so that $T^\epsilon(0) = \log \epsilon$ and $T(1) = 1$. From the perspective of the almost complex structure, J^ϵ has the effect of replacing our neighborhood $(0,1) \times S$ with $(\log \epsilon, 1) \times S$, where the almost complex structure on this lengthened cylinder with coordinate T is just the symmetric extension of the old J.

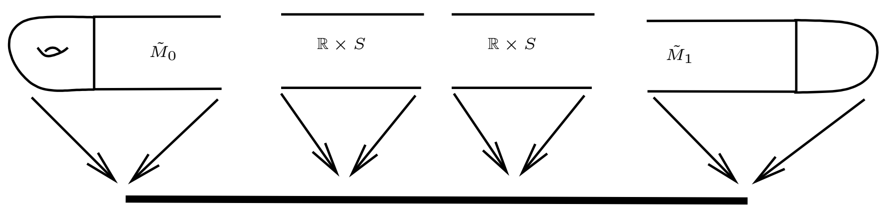


A short calculation shows that J^ϵ is still tamed by ω. Note that we have a great amount of flexibility in how the almost complex manifold (M, J^ϵ) wears the symplectic form ω. Different choices of T^ϵ will concentrate ω in different regions of the cylinder. This is important for being able to tame holomorphic curves in the limit as $\epsilon \to 0$.

Suppose that this hypersurface S separates M into M_0 and M_1. In the limit as $\epsilon \to 0$, our almost complex manifold becomes modelled on the manifold with cylindrical ends $\tilde{M}_0$ obtained by gluing $(0,\infty) \times S$ to M_0, the manifold $\tilde{M}_1$ obtained by gluing $(-\infty, 1) \times S$ to M_1, or the cylinder $\mathbb{R} \times S$. In some sense, the 'limit' of this family of almost complex manifolds should include an infinite number of these cylindrical pieces, because one can construct a sequence of maps of $\mathbb{Z}$ to (M, J^ϵ) so that in the limit, the model around the image of each point is a cylinder which contains none of the other points.

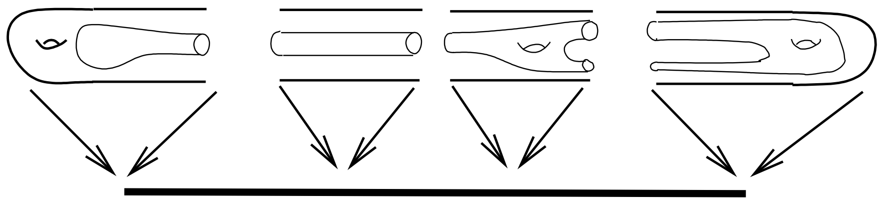


We will be able to view this family (M, J^ϵ) as part of a connected 'smooth' family of exploded fibrations which contains an object that has a base equal to the interval $[0,1]$ and fibers equal to $\tilde{M}_0$ over 0, $\tilde{M}_1$ over 1 and a copy of $\mathbb{R} \times S$ over each point in the interior.



In section 2, we will describe the exact structure on these fibers, in section 3, we will describe the structure on the base, and in section 4, we will say exactly how these fit together. The category of exploded fibrations has a well defined product, so we will also be able to deal with degenerations which look locally like products of degenerations of this type.

If the dynamics of X_t satisfy a certain condition (which holds generically), the limit of J^ϵ holomorphic curves under this degeneration look like a number of holomorphic curves mapping into the different model fibers which are asymptotic to cylinders over orbits of X_t. The holomorphic curves inside all but a finite number of the cylindrical models over the interior of the interval consist just of cylinders over orbits of X_t which coincide with the asymptotics of the next nontrivial holomorphic curves in models to the left or right.



Such holomorphic curves in exploded fibrations are morphisms in the exploded category with a slightly weaker 'smooth' structure. A special type of exploded fibration where holomorphic curves are better behaved is called an exploded $\mathbb{T}$ fibration. The degeneration above would give an exploded $\mathbb{T}$ fibration if the action of X_t was a free circle action. We will define the category of exploded $\mathbb{T}$ fibrations in section 5, and sketch the theory of holomorphic curves in exploded $\mathbb{T}$ fibrations in section 6.

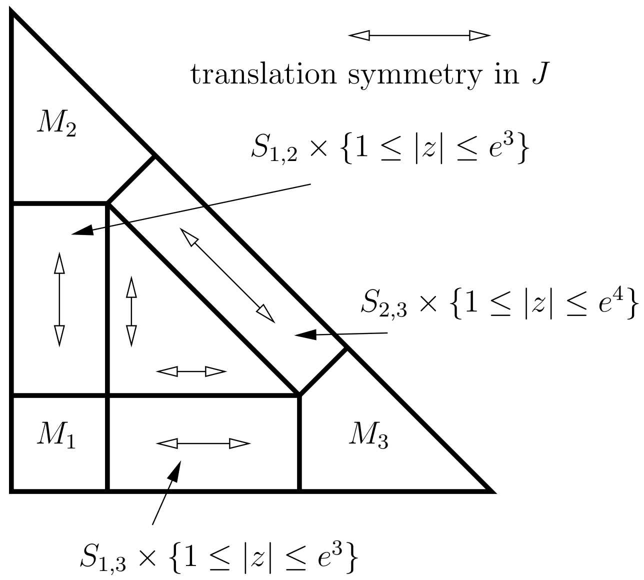
For a second example, consider $\mathbb{C}P^2$ with the standard coordinates $[z_0, z_1, z_2]$, and an action of $\mathbb{T}^2$ by $[z_0, z_1, z_2] \mapsto [z_0, e^{\theta_1 i} z_1, e^{\theta_2 i} z_2]$. We can choose a symplectic form taming the complex structure on $\mathbb{C}P^2$ which is preserved by this $\mathbb{T}^2$ action. Such an ω is of the form $dh_1 \wedge d\theta_1 + dh_2 \wedge d\theta_2$ where h_1 and h_2 are $\mathbb{T}^2$ invariant functions. The functions h_1 and h_2 are the Hamiltonian functions generating the action of θ_1 and θ_2. The image of (h_1, h_2) is called the moment polytope. In the case of $\mathbb{C}P^2$, we can choose this to be the triangle $\{0 \leq h_1, 0 \leq h_2, 5 \geq h_1 + h_2\}$. There is a large amount of flexibility in how we choose h_1 and h_2. In particular, we can make the following choices

(1) $h_1 = \log\left|\frac{z_1}{z_0}\right|$ and $h_2 = h_2\left(\left|\frac{z_2}{z_0}\right|\right)$ on $\{1 \leq h_1 \leq 3, h_1 + h_2 \leq 4\}$

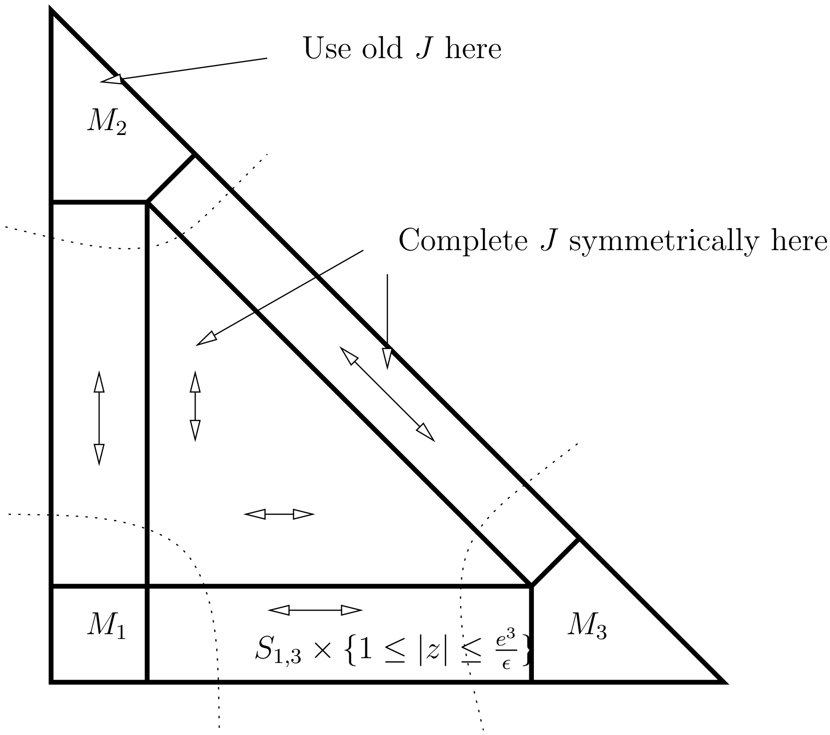
(2) $h_2 = \log\left|\frac{z_2}{z_0}\right|$ and $h_1 = h_1\left(\left|\frac{z_1}{z_0}\right|\right)$ on $\{1 \leq h_2 \leq 3, h_1 + h_2 \leq 4\}$

(3) $h_1 - h_2 = \log\left|\frac{z_1}{z_2}\right|$ and $h_1 + h_2 = (h_1 + h_2)\left(\left|\frac{z_0^2}{z_1 z_2}\right|\right)$ on $\{-2 \leq h_1 - h_2 \leq 2, h_1 + h_2 \geq 4\}$

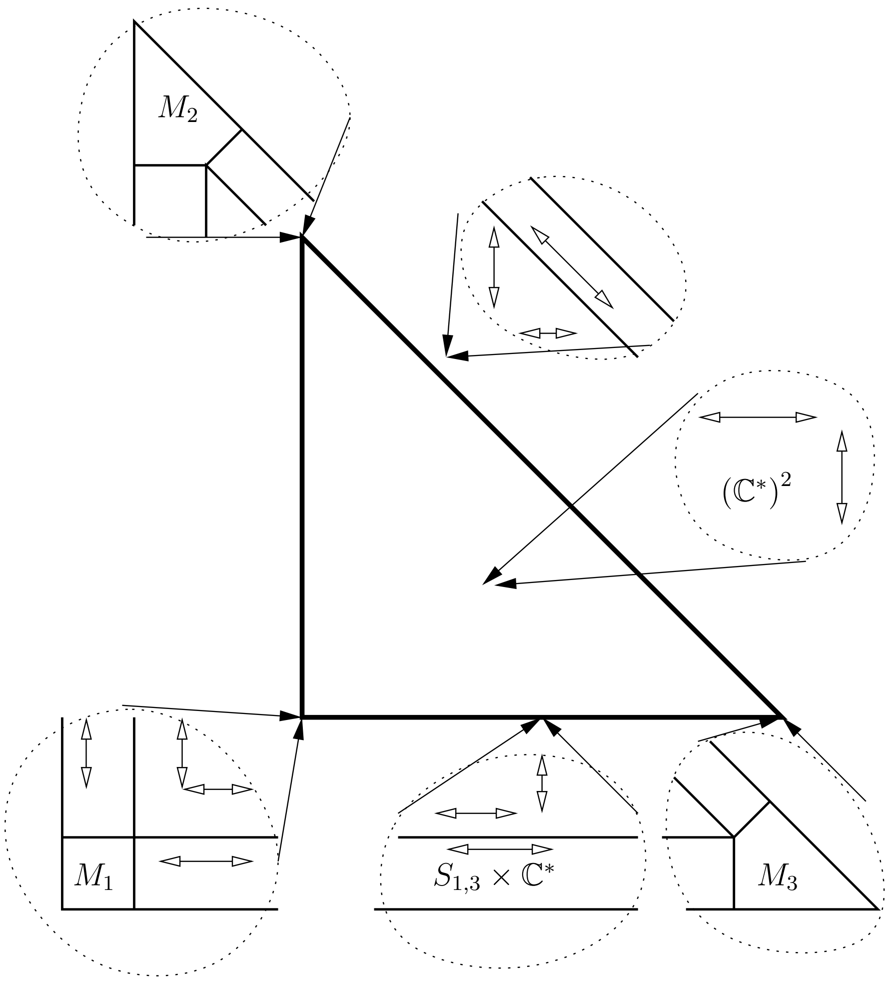
This gives a complex structure with the symmetry over the moment polytope shown in the figure below.



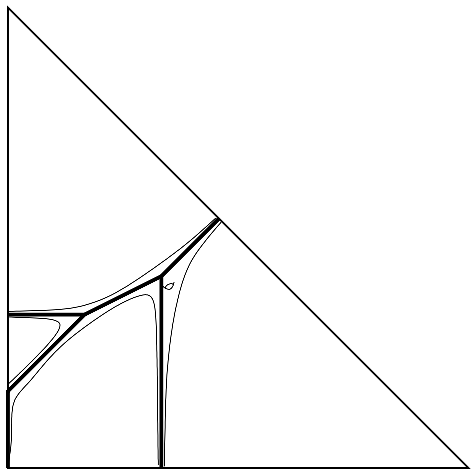
We will now draw a degeneration of complex structure that could also be applied if we glued anything into the three regions M_1, M_2 and M_3 in the above picture. As in the previous example, there is a large amount of flexibility in how this new complex manifold wears the symplectic structure. We choose some way which doesn't change in the regions M_1, M_2 and M_3.



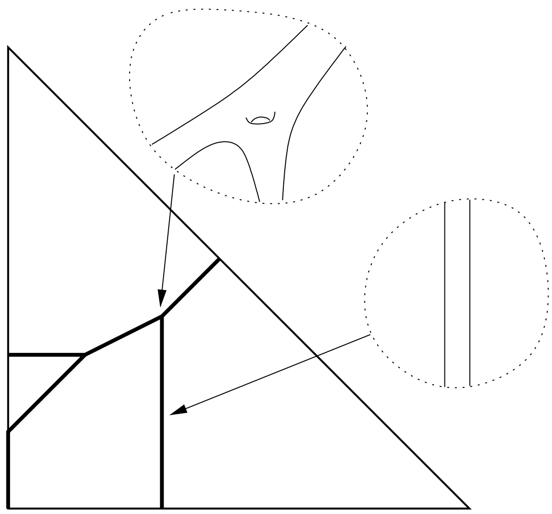
Imagine rescaling this triangle to keep it a constant size as we change J. In the limit, this will give a base for a limiting exploded $\mathbb{T}$ fibration. The fiber over each point in this base will be the local model for what the complex structure looks like around this point in the limit.



It follows from chapter 3 in [3] that in this limit, the image in the above diagrams of holomorphic curves with bounded energy and genus will converge to piecewise linear graphs. Moreover, holomorphic curves converge locally to a collection of holomorphic maps into the local models which are the fibers in the limiting exploded $\mathbb{T}$ fibration.



The diagram below shows a limiting holomorphic curve, with its image in the base, and schematic pictures of the holomorphic curves in the local model fibers over the base.



The rest of this paper will be devoted to rigorously defining exploded fibrations and sketching some of their properties. This will not deal with compactness theorems for holomorphic curves and the related questions of how symplectic forms are used to tame holomorphic curves in exploded fibrations.

2. Log smooth structures and the normal neighborhood bundle

The fibers of exploded fibrations will be log smooth spaces.

One way of thinking of a log smooth manifold is a manifold M with boundary and corners along with the sheaf of vector fields $C^\infty({}^{\log}TM)$ which are tangent to the boundary and corner strata. By manifold with boundary and corners, we mean a manifold with coordinate charts modeled on open subsets of $[0,\infty)^n$. By smooth functions in such a coordinate chart, we mean functions which extend to smooth functions on $\mathbb{R}^n \supset [0,\infty)^n$. The sheaf of vector fields tangent to boundary and corner strata is equal to smooth sections of a vector bundle ${}^{\log}TM$ which we will call the log tangent space. A basis for this vector bundle in the above coordinate chart is give by $\{x_i\frac{\partial}{\partial x_i}\}$. This is often called the b tangent space. The natural objects defined on a manifold with such a structure are smooth sections of tensor products of ${}^{\log}TM$ and its dual, ${}^{\log}T^*M$. These tensor fields have asymptotic symmetry as they approach boundary and corner strata in the sense that the Lie derivative of a smooth tensor field in the direction of $x_i\frac{\partial}{\partial x_i}$ approaches 0 like x_i. Analysis on manifolds with this structure has been studied extensively by Melrose and others. I wish to thank Pierre Albin for explaining this approach to me. The structure we will describe on log smooth manifolds is the smooth analogue of log structures in algebraic geometry.

The stratified structure of M will be important. Call each connected component of the boundary of M a boundary strata. We will consider this strata to be an immersed submanifold $\iota_S : S \longrightarrow M$ with boundary and corners, which is embedded on the interior of S. The other strata of M consist of M itself, and the connected components of higher codimensional corner strata, which are again considered to be immersed submanifolds $\iota_S : S \longrightarrow M$ which are embedded on their interiors.

We will think of log smooth manifolds as possibly non compact manifolds with an idea of a smooth structure at infinity so that every boundary strata S corresponds to a cylindrical end at infinity $\mathcal{N}_SM$, called the normal neighborhood bundle of S.

Definition 2.1. Given an open subset $U \subset M$ and an open set $U_S \subset S$ inside a boundary strata $\iota_S : S \longrightarrow M$, a boundary defining function on U defining U_S is a smooth function $x : M \longrightarrow [0,\infty)$ so that

$$\{x^{-1}(0)\} = \iota_S(U_S) \subset M$$

and dx is nonvanishing near S.

Definition 2.2. Given an open set $U \subset M$, a log C^k function $f \in {}^{\log}C^k(U)$ is a function defined on the interior of U so that

$$f := g + \sum \alpha_i \log(x_i)$$

where x_i are boundary defining functions on U, $\alpha_i \in \mathbb{Z}$, and $g \in C^k(U)$.

This defines a sheaf on M of log C^k functions, which we denote by ${}^{\log}C^k(M)$

Definition 2.3. A log C^k map between log smooth manifolds

$$f : M \longrightarrow N$$

is a continuous map f so that the pull back of the sheaf of log C^k functions on N is contained in the sheaf of log C^k functions on M.

$$f^*\left({}^{\log}C^k(N)\right) \subset {}^{\log}C^k(M)$$

We want to now enlarge the log smooth category to include integral affine bundles over log smooth manifolds.

Definition 2.4. An integral affine structure on a manifold M^n is a $\mathbb{Z}^n$ lattice inside T_xM for each $x \in M$ which is preserved by some flat connection.

(By an integral $\mathbb{Z}^n$ lattice, we mean the image of some proper homomorphism $\mathbb{Z}^n \longrightarrow T_xM$.)

An integral vector in T_xM is a vector in this integral lattice. A basic vector in T_xM is an integral vector which is not a multiple of another integral vector.

An integral affine map $f : M^m \longrightarrow N^n$ between integral affine manifolds is a smooth map so that df sends integral vectors to integral vectors.

We will regard a log smooth function as a log smooth map from M to $\mathbb{R}$ with the standard integral affine structure given by the lattice $\mathbb{Z} \subset T\mathbb{R}$. A log smooth map $f : M \longrightarrow \mathbb{R}^k$ with the standard integral affine structure given by $\mathbb{Z}^k \subset T\mathbb{R}^k$ is given by $(f_1, \ldots, f_k)$ where $f_i \in {}^{\log}C^\infty(M)$.

Definition 2.5. An integral affine $\mathbb{R}^k$ bundle M over a log smooth manifold S is a fibration

$$\begin{array}{ccc} \mathbb{R}^k & \longrightarrow & M \\ & & \downarrow \pi \\ & & S \end{array}$$

so that there exists an open cover U_α of S so that $\pi^{-1}(U_\alpha) = \mathbb{R}^k \times U_\alpha$ and transition functions on $U_\alpha \cap U_\beta$ are of the form

$$(x, s) \mapsto (A(x) + f(s), s)$$

where $f : U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^k$ is a log smooth map and A is an integral affine automorphism $\mathbb{R}^k \longrightarrow \mathbb{R}^k$.

We will call integral affine bundles over log smooth manifolds log smooth spaces. A strata of a log smooth space is the pullback of this bundle to a strata of the base.

Definition 2.6. A log C^k function $f \in {}^{\log}C^k(M)$ on a log smooth space M which is an integral affine $\mathbb{R}^n$ bundle is a function defined on the interior of M so that in any local product neighborhood $\mathbb{R}^n \times U \subset M$

$$f(x,u) := A(x) + g(u)$$

where $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ is given by an integer matrix and $g \in {}^{\log}C^k(U)$.

Definition 2.7. A log C^k map between log smooth spaces

$$f : M \to N$$

is a continuous map from the base of M to the base of N which lifts to a map from the interior of M to the interior of N so that the pull back of the sheaf of log C^k functions on N is contained in the sheaf of log C^k functions on M.

$$f^* \left({}^{\log}C^k(N)\right) \subset {}^{\log}C^k(M)$$

A log smooth vector field on a log smooth space, $v \in C^\infty\left({}^{\log}TM\right)$ is a smooth vector field which is tangent to the boundary strata and invariant under local affine translation of the fibers. These are smooth sections of the log tangent bundle ${}^{\log}TM$ invariant under local affine translation. Log smooth isotopys of M are generated by the flow of log smooth vector fields.

The fact that the flow of log smooth vector fields preserves boundary strata has the consequence that we can not naturally identify a neighborhood of a boundary strata with the log normal bundle. Instead, we have a naturally defined affine bundle called the normal neighborhood bundle which has an action of the log normal bundle on it.

Definition 2.8. The normal neighborhood bundle $\mathcal{N}_sM$ over a point $s \in M$ is the space of functions ν_s called evaluations

$$\nu_s : {}^{\log}C^\infty(M) \longrightarrow \mathbb{R}$$

so that

$$\nu_s(f) = f(s) \text{ if } f(s) \text{ is defined}$$
$$\nu_s(f+g) = \nu_s(f) + \nu_s(g)$$

If s is in a codimension k boundary or corner strata given locally by the vanishing of boundary defining functions $x_1, \dots, x_k$, then $\mathcal{N}_sM$ is an integral affine space equal to $\mathbb{R}^k$. This identification is given by $\nu \mapsto (\nu(x_1), \dots, \nu(x_k))$.

Any log smooth function f induces a function on the normal neighborhood bundle called its restriction by $f(\nu_s) := \nu_s(f)$.

Definition 2.9. The normal neighborhood bundle of a codimension k strata $S \subset M$, $\mathcal{N}_S(M)$ is a log smooth $\mathbb{R}^k$ bundle over S with interior given by evaluations $\nu_s \in \mathcal{N}M$ so that s is in the interior of S. The sheaf of log smooth functions on $\mathcal{N}_SM$ is given by the restriction of the sheaf of log smooth functions on M in the sense that f is log smooth on $U \subset \mathcal{N}_S M$ if there exists an open set $\tilde{U} \subset M$ which contains the projection of U to S and a log smooth function $\tilde{f}$ defined on U which restricts to f.

Definition 2.10. An outward direction in $\mathcal{N}_S M$ is a direction in which the evaluation of any boundary defining function for S decreases.

This is called 'outward' because if we use $N_S M$ to give coordinates on a neighborhood of $S \subset M$, these directions move out towards the strata S which we consider to be out at infinity.

When all strata $\iota_S : S \longrightarrow M$ are injective, $\mathcal{N}_S M$ is easier to describe. In this case, S is defined by k boundary defining functions $x_1 \dots x_k$ and $\mathcal{N}_S M$ is equal to $\mathbb{R}^k \times S$ with coordinates $(\nu_1, \dots, \nu_k, s)$. The restriction of $\log(x_i)$ to $\mathcal{N}_S M$ is given by the coordinate function ν_i. This means that for $g \in {}^{\log}C^\infty(M)$ defined on the interior of S, the restriction of $g(x, s) + \sum \alpha_i \log(x_i)$ is given by $g(0, s) + \sum \alpha_i \nu_i$. An outward direction in this situation would be any direction that preserves the coordinate s and is negative in the first k coordinates.

One trivial example of a normal neighborhood bundle is $\mathcal{N}_M M = M$.

Lemma 2.1. *Given a log smooth map $f : M \longrightarrow N$, there is a natural map of normal neighborhood bundles*

$$f : \mathcal{N}M \longrightarrow \mathcal{N}N$$

defined by

$$f(\nu)(g) := \nu(g \circ f)$$

Also, if the interior of a strata $S \subset M$ is sent to the interior of a strata $S' \subset M$, then this is a natural map called the restriction of f to $\mathcal{N}_S M$ which is a log smooth map

$$f : \mathcal{N}_S M \longrightarrow \mathcal{N}_{S'} N$$

Proof. The naturality of this map is clear from the definition. We must check that the induced map $f : \mathcal{N}_S M \longrightarrow \mathcal{N}_{S'} N$ is log smooth.

First, note that there is a unique continuous map $S \longrightarrow S'$ which is equal to f on the interior of S. This gives the continuous map on the base of $\mathcal{N}_S M$. To see that the pullback of log smooth functions on $\mathcal{N}_{S'} N$ are log smooth, note that they come locally from the restriction of log smooth functions $g \in \mathcal{C}^\infty(N)$, so our function is locally $\nu(g)$. Pulling this back via f gives the function $\nu(g \circ f)$. But $f : M \longrightarrow N$ is log smooth, so $g \circ f$ is log smooth, and therefore the restriction $\nu(g \circ f)$ must also be log smooth.

□

In particular, the action on M given by the flow of a log smooth vector field lifts to an action on $\mathcal{N}M$. Differentiating this action gives a lift of log vector fields M to vector fields on $\mathcal{N}M$. This lift preserves the Lie bracket, addition of vector fields and multiplication by functions. It sends nonzero log smooth vector fields to nonzero vector fields.

Definition 2.11. A log smooth morphism

$$f : M \to N$$

is a log smooth map

$$f : M \to \mathcal{N}_S M$$

for some strata $S \subset M$.

This finally defines the category of log smooth spaces. We will keep the distinction between log smooth maps and log smooth morphisms.

3. Stratified integral affine spaces

The base of our exploded fibration will be a stratified integral affine space.

Definition 3.1. A stratified integral affine space is a finite category B associated to a topological space $|B|$ with the following structure

(1) Objects are integral affine strata B_j^k with integral affine boundary and corners, so that each point in B_j^k has a neighborhood equal to an open set in $\mathbb{R}^m \times [0, \infty)^{k-m}$ with the standard integral affine structure.

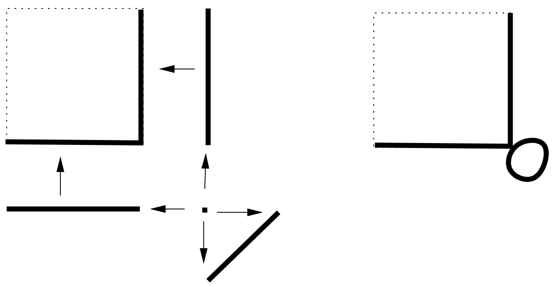
(2) Morphisms are integral affine inclusions

$$\iota : B_i^m \hookrightarrow B_j^k$$

so that each connected affine boundary or corner strata of B_j^k is the image of a unique inclusion. These inclusions and the identity maps are the only morphisms.

(3) The space $|B|$ is the disjoint union of strata quotiented out by all inclusions.

$$|B| := \frac{\coprod B_j^k}{\iota : B_i^m \hookrightarrow B_j^k}$$



B $|B|$

Definition 3.2. A stratified integral affine map from one stratified integral affine space to another

$$f : B \longrightarrow C$$

consists of

(1) A functor

$$F : B \longrightarrow C$$

(2) Integral affine maps

$$f : B_j^k \longrightarrow F(B_j^k)$$

so that the interior of strata in B is sent to the interior of a strata in C and so that

$$f \circ \iota := F(\iota) \circ f$$

This induces a map on the underlying topological spaces

$$f : |B| \longrightarrow |C|$$

4. Exploded fibrations

Definition 4.1. An exploded fibration $\mathfrak{B}$ has the following structure.

(1) A stratified integral affine base B

(2) A fibration over each strata $\mathfrak{B}_j^k \longrightarrow B_j^k$ with log smooth fibers

$$M_b := T_b B_j^k \times W_b^{n-k}$$

which are integral affine $T_b B_j^k$ bundles. The identification of M_b as a product is not canonical, however the action of $T_b B_j^k$ is well defined.

(3) A flat connection on each fibration $\mathfrak{B}_j^k$ so that parallel transport gives log smooth isomorphisms compatible with the action of TB_j^k and affine parallel transport.

(4) For each inclusion ι in B, there is a dual log smooth morphism,

$$\iota^\dagger : M_{\iota(b)} \longrightarrow M_b$$

so that

(a) For some strata $S \subset M_b$, $\iota^\dagger$ is a log smooth isomorphism

$$\iota^\dagger : M_{\iota(b)} \longrightarrow \mathcal{N}_S M_b$$

(b) The induced action of $T_{\iota(b)} B$ on $\mathcal{N}_S M_b$ is compatible with $d\iota$ in the sense that the induced action of $d\iota(v)$ is equal to the action of v. The action of a vector in $T_{\iota(b)} B$ which points towards the interior of the strata moves $\mathcal{N}_S M_b$ outwards.

(c) Every strata of M_b is the image of a unique isomorphism $\iota^\dagger$.

(d) $(\iota_1 \circ \iota_2)^\dagger = \iota_2^\dagger \circ \iota_1^\dagger$

(e) The isomorphisms $\iota^\dagger$ are compatible with parallel transport.

Note that this definition assumes that the normal neighborhood bundles $\mathcal{N}_S M_b$ are trivial. This is a global condition. If it is removed, then the base becomes an orbifold version of a stratified integral affine space and the fibration over each strata is a kind of Seifert fibration.

Definition 4.2. A morphism $f : \mathfrak{B} \longrightarrow \mathfrak{C}$ is given by

(1) A stratified integral affine map of bases $f : B \longrightarrow C$.
(2) Log smooth maps
$$f_b : M_b \longrightarrow M_{f(b)}$$
   (a) equivariant with respect to the action of $T_b B$ and df
   (b) compatible with the isomorphisms $\iota^\dagger$ in the sense that
$$f_b \circ \iota^\dagger = F(\iota)^\dagger \circ f_{\iota(b)}$$
   (c) compatible with parallel transport within strata.

Definition 4.3. A morphism $f : \mathfrak{B}' \longrightarrow \mathfrak{B}$ is a refinement if

(1) the map on bases $f : |B'| \longrightarrow |B|$ is a homeomorphism and df is an integral affine isomorphism onto its image
(2) the maps of fibers $f_b : M_b \longrightarrow M_{f_b}$ are diffeomorphisms of the interior of M_b to the interior of M_{f_b}.

4.1. Coordinates

Definition 4.4. A log smooth coordinate chart on a log smooth space M which is an integral affine $\mathbb{R}^m$ bundle is an open subset $U \subset M$ and log smooth functions $x_i \in {}^{\log}C^\infty(U)$, so that

(1) e^{-x_i} give smooth coordinate functions for U.
(2) $\{dx_1, \dots, dx_m\}$ restricted to $\mathbb{R}^m$ fibers provide a dual basis for the integral affine structure. Call these the affine coordinate functions.
(3) x_i for $i > m$ are $\mathbb{R}^m$ invariant.
(4) $\{e^{-x_{m+1}}, \dots, e^{-x_{m+k}}\}$ are boundary defining functions which define the boundaries of U. Call x_i boundary coordinate functions for $m < i \leq m + k$.
(5) x_i for $i > m + k$ are called the smooth coordinate functions.

Another way to say this is that log smooth coordinates on U are induced by a log smooth map $U \longrightarrow \mathbb{R}^m \times ([0, \infty)^k \times \mathbb{R}^n)$ which is an isomorphism onto its image. The coordinate functions of $\mathbb{R}^m$ give the affine coordinates, the boundary coordinate are given by $-\log$ of the coordinate functions of $[0, \infty)^k$, and the smooth coordinate functions are the coordinates of $\mathbb{R}^n$.

The following lemma follows directly from the definition of log smooth maps.

Lemma 4.1. *A map defined on the interior of a log smooth space is log smooth if and only if the following hold*

(1) *The pull back of affine coordinate functions are log smooth functions, which means that they are integral sums of affine and boundary coordinate functions and a smooth function depending on smooth coordinates and boundary defining functions.*
(2) *The pull back of any boundary coordinate function consists of a positive integral sum of boundary coordinate functions, and a smooth function depending on smooth coordinates and boundary defining functions.*
(3) *The pull back of any smooth coordinate function consists of a smooth function depending on smooth coordinates and boundary defining functions.*

Given a log smooth coordinate chart over U we can canonically associate an exploded fibration called the explosion of U, $\text{Expl}(U)$. The base of $\text{Expl}(U)$ is a cone inside the integral affine space $\mathbb{R}^{m+k}$ given by the subset where $y_i \geq 0$ for $m < i \leq m+k$. Let $I \subset \{m+1, \dots, m+k\}$ be some subset of the boundary indices and S_I the strata defined by $\{e^{-x_i} = 0, i \in I\}$. The fibers over the set where $\{y_i > 0, i \in I\}$ is $\mathcal{N}_{S_I}U$. We can restrict our log smooth coordinate chart on U to $\mathcal{N}_{S_I}U$. This changes the boundary coordinates x_i into affine coordinates for $i \in I$. The action of a base tangent vector $\sum \alpha_i \frac{\partial}{\partial y_i}$ on a fiber is given in these coordinates by $x_i \mapsto x_i + \alpha_i$.

The construction of $\text{Expl}(U)$ is natural in the sense that given any log smooth map $U_1 \longrightarrow U_2$ there is a natural map $\text{Expl}(U_1) \longrightarrow \text{Expl}(U_2)$. This tells us that there is a natural construction of of $\text{Expl}(M)$ for any log smooth manifold M.

Definition 4.5. The exploded fibration $\mathfrak{R}^n$ is defined as the standard fibration with base $\mathbb{R}^n$ and fibers all equal to $\mathbb{R}^n$.

Definition 4.6. An exploded coordinate chart $\mathfrak{U}$ on $\mathfrak{B}$ is a pair of smooth exploded morphisms

$$\begin{array}{l}\mathfrak{U} \longrightarrow \mathfrak{B} \\ \downarrow \\ \mathfrak{R}^n\end{array}$$

which, when restricted to any fiber of $\mathfrak{B}$ give a log smooth coordinate chart in a suitable choice of basis for $\mathfrak{R}^n$.

Definition 4.7. An exploded fibration $\mathfrak{B}$ is covered by exploded coordinate charts $\mathfrak{U}_\alpha$ if the log smooth coordinate charts obtained by restricting $\mathfrak{U}_\alpha$ cover every fiber.

Note that things defined by patching together exploded coordinate charts may fail to have the global property that the fibers are trivial affine bundles. Otherwise, they will always be exploded fibrations.

4.2. Refinements and blowups

In this section, we will show that any subdivision of the base of an exploded fibration defines a unique refinement. This will also be used later when we explore the intersection theory of exploded fibrations. The construction will be natural, so we will work in local coordinate charts on the fibers.

Given the explosion Expl(U) of some log smooth coordinate chart, we will now construct a refinement Expl($\tilde{U}$) $\longrightarrow$Expl(U) for any integral subdivision of the base into a union of cones. We will use this as a local model for refinements.

A cell of this subdivision will be of the form $\{y \cdot \alpha_j \geq 0\}$ for some collection of α_j which form an integral basis for some subspace $\mathbb{R}^{k+l} \subset \mathbb{R}^{m+k}$. Any other cell is given by $\{y \cdot \alpha'_j \geq 0\}$ where $\{\alpha'_j\}$ form a different basis for the same subspace $\mathbb{R}^{k+l} \subset \mathbb{R}^{m+k}$. Each new cell will correspond to a new log smooth coordinate chart $U^{\{\alpha_i\}}$ with coordinates

$$\{x \cdot \beta_1, \dots, x \cdot \beta_{m-l}, x \cdot \alpha_1, \dots, x \cdot \alpha_{k+l}, x_{m+k+1}, \dots\}$$

Here $\{\beta_i, \alpha_j\}$ form an integral basis for $\mathbb{R}^{m+k}$. $x \cdot \beta_i$ are the affine coordinate functions, $x \cdot \alpha_i$ are the boundary coordinate functions, and the remaining x_i are the smooth coordinate functions.

The domain of definition of these coordinates, $U^{\{\alpha_i\}}$, is considered to extend to boundaries where $e^{-x \cdot \alpha_i} = 0$. In essence, the new log smooth coordinates are the old ones with a relabeling of what is considered to be a boundary coordinate function or affine coordinate function. This operation is a blowup which creates new boundary components. Each new edge $\{\alpha_j = 0, \text{ for } j \neq i\}$ of a cell $\{\alpha_j \cdot x \geq 0\}$ corresponds to a new boundary component so that the boundary coordinate function is given by $\alpha_i \cdot x$ and the remaining coordinate functions give coordinates on the boundary.

The intersection of two of these new coordinate charts include the interiors of the charts and the boundaries corresponding to common edges. The restriction of log smooth coordinates to this intersection changes the role of the other boundary defining functions to smooth coordinate functions. A transition map on this intersection between two sets of coordinates then sends affine coordinate functions to affine coordinate functions, boundary coordinate functions to boundary defining functions plus some smooth coordinate functions, and smooth coordinate functions to integral combinations of smooth coordinate functions. As such they are log smooth. Patching these coordinate charts together gives a log smooth blowup $\tilde{U} \longrightarrow U$. Due to the naturality of the explosion construction, this gives a refinement Expl($\tilde{U}$) $\longrightarrow$Expl(U).

Lemma 4.2. *Given any refinement $f : \mathfrak{B}' \longrightarrow \mathfrak{B}$, any log smooth coordinate chart U on a fiber M_{f_b} of $\mathfrak{B}$ lifts as above to $\tilde{U} = f_b^{-1}(U)$.*

Proof. We need to show that for any log smooth coordinate chart U on $M_{f(b)}$, $U^{\{\alpha_i\}}$ gives a log smooth coordinate chart on $f_b^{-1}(U) \subset M_b$, where $U^{\{\alpha_i\}}$ is the blowup of coordinate charts given by a cell of the subdivision of the base in the notation above. Moreover, we need to show that the collection of sets $U^{\{\alpha_i\}}$ from all cells covers $f_b^{-1}U$.

The proof is by induction. This holds for fibers that have no boundary coordinates, as then f is a diffeomorphism and preserves the integral affine structure on fibers, so it is a log smooth isomorphism, and $U^{\{\alpha_i\}} = U$.

Suppose that the lemma holds for fibers of $\mathfrak{B}'$ with strata of codimension less than k.

Let M_b be a fiber of $\mathfrak{B}'$ with strata of codimension less than or equal to k, and let U be a coordinate chart on $M_{f(b)}$. The subdivision of the base B gives a subdivision of the base of $\mathrm{Expl}(U)$. Choose a cell of this subdivision given by $\{\alpha_i \geq 0\}$. We must show that $U^{\{\alpha_i\}}$ gives a log smooth coordinate chart on M_b.

Let I be a proper subset of the indexes of $\{\alpha_i\}$, $S_I \subset M_b$ and ι_I the corresponding strata and base inclusion. Denote the restriction of U to $\mathcal{N}_{f(S_I)} M_{f(b)}$ by U_I. $\mathcal{N}_{S_I} M_b = M_{\iota_I(b)}$ only has strata of codimension less than k so the inductive hypothesis holds and $U_I^{\{\alpha_i, i\in I\}}$ is a log smooth coordinate chart on $\mathcal{N}_{S_I} M_b = M_{\iota_I(b)}$. This chart $U_I^{\{\alpha_i, i\in I\}}$ is also the restriction of $U^{\{\alpha_i\}}$ to $\mathcal{N}_{S_I} M_b$, so we have that the restriction of $U^{\{\alpha_i\}}$ to the normal neighborhood bundle of any boundary strata gives log smooth coordinates. We also have that the restriction of all such $U^{\{\alpha_i\}}$ to normal neighborhood bundles covers the restriction of $f_b^{-1}(U)$ to the normal neighborhood bundle. Therefore the sets $U^{\{\alpha_i\}}$ must cover $f^{-1}(U)$ because each one covers the interior, and together they cover the boundary strata.

Let's now check that the axioms of a log smooth coordinate chart are satisfied by $U^{\{\alpha\}}$. First, all coordinates are log smooth functions because f is a log smooth map, and the coordinates are just pullbacks of integral sums of log smooth functions.

Item 2 is satisfied, as our choice of fiber coordinate functions is just the pullback of a subset of the fiber coordinate functions on the target, f_b is equivariant with respect to the action of the tangent space of the base and df, and df is an integral affine isomorphism onto its image.

Item 3 is satisfied. The boundary coordinates $x \cdot \alpha_i$ are constant on fibers because f_b is equivariant and the α_i are orthogonal to the fiber directions. The smooth coordinates are just pullbacks of smooth coordinates, so they too are constant on fibers.

Item 4 is satisfied. $e^{-x\cdot\alpha_i}$ is a boundary defining function because it is log smooth, and restricts correctly to the normal neighborhood of the boundary it defines.

Item 1 is satisfied. This is because the coordinate functions are log smooth, give smooth coordinates on the interior of $U^{\{\alpha_i\}}$, their restrictions to normal neighborhood bundles give smooth coordinate functions, and $e^{-x\cdot\alpha_i}$ are boundary defining functions.

Item 5 is an empty condition, so $U^{\{\alpha_i\}}$ is a log smooth coordinate chart and the lemma is proved.

□

Proposition 4.3. *Given a morphism $f : \mathfrak{B} \to \mathfrak{C}$ and a refinement $\mathfrak{C}' \longrightarrow \mathfrak{C}$ so that the map of bases $f : B \to C$ lifts to a map $\tilde{f} : B \longrightarrow C'$, there exists a unique lift $\tilde{f} : \mathfrak{B} \longrightarrow \mathfrak{C}'$ of f to a morphism to $\mathfrak{C}'$.*

Proof. Refinements are diffeomorphisms on the interiors of fibers, and log smooth maps are determined by their restriction to the interior, so uniqueness is automatic. We just need to check that the maps given by $f_b = \tilde{f}_b$ restricted to the interior of M_b are log smooth and satisfy the compatibility requirements for exploded fibrations.

To show that $\tilde{f}_b$ is log smooth, we use the criteria from Lemma 4.1 and the local normal form for refinements from Lemma 4.2.

Affine coordinates on the blowup $\tilde{M}_c$ are a subset of affine coordinate on M_c, so criterion 1 is automatically satisfied. Smooth coordinates for $\tilde{M}_c$ in the normal form from Lemma 4.2 are just smooth coordinates from M_c, so criterion 3 is also automatically satisfied, so we just need to study the pull back of boundary coordinate functions.

The fact that f is log smooth tells us the following: the pull back under $\tilde{f}$ of boundary coordinate functions on $\tilde{M}_c$ are integral sums of affine and boundary coordinates plus a smooth function depending on boundary defining functions and smooth coordinates. The exact combination of affine and boundary coordinates is determined by the map $f : B \longrightarrow C$ on bases. The requirement that f lifts to the subdivision C' is equivalent to this combination being a positive combination of boundary coordinates, which tells us that $\tilde{f}$ satisfies criterion 3, and is therefore log smooth.

The fact that $\tilde{f}$ is compatible with parallel transport follows from the fact that f is compatible and $\tilde{f}$ and f are equal on the interiors of fibers. Also, restricted to interiors, $\tilde{f} \circ \iota^\dagger = f \circ \iota^\dagger = F(\iota)^\dagger \circ f = \tilde{F}(\iota)^\dagger \circ \tilde{f}$, so $\tilde{f}$ is also compatible with the inclusions $\iota^\dagger$ and f is an exploded morphism.

□

Theorem 4.4. *Given an exploded fibration $\mathfrak{B}$ and a stratified integral affine map*

$$f : B' \longrightarrow B$$

so that f is a homeomorphism and df is an integral affine isomorphism onto its image, there exists a unique refinement $f : \mathfrak{B}' \longrightarrow \mathfrak{B}$ with base $f : B' \to B$.

Proof. First, Proposition 4.3 tells us that this refinement is unique up to isomorphism.

To show the existence of a refinement, note that any exploded fibration is locally modelled on $\text{Expl}(U)$ for some log smooth coordinate chart U. Given a subdivision, we constructed a refinement $\text{Expl}(\tilde{U}) \longrightarrow \text{Expl}(U)$. This is a local model for our refinement. These refinements so constructed must coincide on the intersection of their domains of definition due to the uniqueness of refinements from Proposition 4.3, so these local refinements patch together to a global refinement.

□

5. Exploded $\mathbb{T}$ fibrations

Exploded $\mathbb{T}$ fibrations can be thought of as exploded fibrations with extra torus symmetry so that instead of fibers being $\mathbb{R}^k$ bundles, fibers are $(\mathbb{C}^*)^k$ bundles. We will first describe the analogue of log smooth spaces in this setting.

5.1. Log smooth $\mathbb{T}$ spaces

A log smooth $\mathbb{T}$ manifold is a connected smooth manifold M locally modeled on $\mathbb{C}^k \times \mathbb{R}^{n-2k}$ with the following sheaf of log smooth $\mathbb{T}$ functions.

Definition 5.1. A log smooth $\mathbb{T}$ function $f \in {}^{\log}_{\mathbb{T}}C^\infty(\mathbb{C}^k \times \mathbb{R}^{n-2k})$ is a map

$$f : (\mathbb{C}^*)^k \times \mathbb{R}^{n-2k} \longrightarrow \mathbb{C}^*$$
$$f := z_1^{\alpha_1} \cdots z_k^{\alpha_k} g$$
$$\text{for } \alpha_i \in \mathbb{Z}, \text{ and } g \in C^\infty(\mathbb{C}^k \times \mathbb{R}^{n-2k}, \mathbb{C}^*)$$

Definition 5.2. A log smooth $\mathbb{T}$ manifold M is a smooth manifold M along with a sheaf of log smooth $\mathbb{T}$ functions ${}^{\log}_{\mathbb{T}}C^\infty(M)$, where if $U \subset M$ is an open subset, $f \in TC^\infty(U)$ is some $\mathbb{C}^*$ valued function defined on a dense open subset of U.

This must satisfy the condition that around any point in M, there exists a neighborhood U and a diffeomorphism $\phi : U \longrightarrow \mathbb{C}^k \times \mathbb{R}^{n-2k}$ so that

$$\phi^*\left({}^{\log}_{\mathbb{T}}C^\infty(\mathbb{C}^k \times \mathbb{R}^{n-2k})\right) = {}^{\log}_{\mathbb{T}}C^\infty(U)$$

Definition 5.3. A smooth function $z : U \subset M \longrightarrow \mathbb{C}$ is a boundary defining function if dz is nonzero when z is zero, and $z \in {}^{\log}_{\mathbb{T}}C^\infty(U)$

Definition 5.4. A strata $S \longrightarrow M$ is a connected properly immersed submanifold of M which is given locally by the vanishing of some number of boundary defining functions, and which is embedded on a dense subset of S.

The interior of a strata S is the strata minus all substrata.

Note that each point in M is in the interior of a unique strata.

With log $\mathbb{T}$ functions defined, other definitions are analogous to the definitions in the log smooth case.

Definition 5.5.

$$f : M \to N$$

is a log C^k $\mathbb{T}$ map if it is continuous, sends the interior of M to the interior of N, and

$$f^*\left({}^{\log}_{\mathbb{T}}C^k(N)\right) \subset {}^{\log}_{\mathbb{T}}C^k(M)$$

Definition 5.6.
A log smooth $\mathbb{T}$ space is a $(\mathbb{C}^*)^n$ bundle

$$\begin{array}{rl} (\mathbb{C}^*)^n \longrightarrow & M \\ & \downarrow \pi \\ & S \end{array}$$

over a log smooth $\mathbb{T}$ manifold S.

In particular, there exists an open cover $\{U_\alpha\}$ of S so that $\pi^{-1}U_\alpha = (\mathbb{C}^*)^n \times U_\alpha$ and transition functions are of the form

$$(z, s) \mapsto (\phi(z)f'(s), s)$$

where

- $f' : U_\alpha \cap U_\beta \longrightarrow (\mathbb{C}^*)^n$ is a product of log smooth $\mathbb{T}$ functions,
- and $\phi : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$ is a group isomorphism.

A $\mathbb{C}^*$ valued function f is in ${}^{\log}_{\mathbb{T}}C^k(\mathbb{C}^* \times S)$ if $f = z^m g$ where $g \in {}^{\log}_{\mathbb{T}}C^k(S)$ and $m \in \mathbb{N}$. Using this inductively defines the sheaf of log smooth $\mathbb{T}$ functions on $(\mathbb{C}^*)^n \times S$, and then using this locally defines the sheaf of log smooth $\mathbb{T}$ functions on a log smooth $\mathbb{T}$ space M which is a nontrivial $(\mathbb{C}^*)^n$ bundle.

Definition 5.7.

$$f : M \longrightarrow N$$

is a log C^k $\mathbb{T}$ map between log smooth $\mathbb{T}$ spaces if it projects to a log C^k $\mathbb{T}$ map on the bases of M and N , and

$$f^*\left({}^{\log}_{\mathbb{T}}C^k(N)\right) \subset {}^{\log}_{\mathbb{T}}C^k(M)$$

Definition 5.8. The normal neighborhood bundle over a point $s \in M$, $\mathcal{N}_s M$ is given by the space of evaluations $\nu_s : {}^{\log}_{\mathbb{T}}C(M) \longrightarrow \mathbb{C}^*$ so that

$$\nu_s(f) = f(s) \text{ if well defined}$$

$$\nu_s(fg) = \nu_s(f)\nu_s(g)$$

Definition 5.9. The normal neighborhood bundle of a codimension $2k$ strata $S \longrightarrow M$ is a log smooth $\mathbb{T}$ space which is a $(\mathbb{C}^*)^k$ bundle over S. The fiber in $\mathcal{N}_S M$ over a point s in the interior of S correspond to evaluations ν_s over s.

If S is locally given by vanishing of boundary defining functions $z_1, \dots, z_k$, then local coordinates for $\mathcal{N}_S M$ are given by $(\nu(z_1), \dots, \nu(z_k), s)$, where the first k coordinates give coordinates for the $(\mathbb{C}^*)^k$ fiber, and s denotes the complimentary coordinates which give coordinates on S.

Definition 5.10. An outward direction in $\mathcal{N}_S M_b$ is a direction in which the restriction of any boundary defining function for S decreases.

Lemma 5.1. *A log $\mathbb{T}$ map $f : M \longrightarrow N$ induces a natural map*

$$f : \mathcal{N}M \longrightarrow \mathcal{N}N$$

defined by

$$f(\nu)(g) = \nu(f \circ g)$$

If f sends the interior of a strata S to the interior of S', then this gives a log $\mathbb{T}$ map

$$f : \mathcal{N}_S M \longrightarrow \mathcal{N}_{S'} N$$

The proof of this lemma is analogous to Lemma 2.1

Definition 5.11. A log $\mathbb{T}$ morphism from M to N is a log $\mathbb{T}$ map

$$f : M \longrightarrow \mathcal{N}_S N$$

for some strata $S \in N$.

If we take the oriented real blowup of all strata of a log smooth $\mathbb{T}$ space, we obtain a log smooth space. Locally, this involves replacing any torus boundary defining function z with a log smooth boundary defining function $|z|$ and a smooth coordinate $\frac{z}{|z|}$. In this way, we can view any log smooth $\mathbb{T}$ space as a log smooth space. Any log $\mathbb{T}$ morphism is still a log morphism, and the normal neighborhood bundle in the log $\mathbb{T}$ setting is the normal neighborhood in the log smooth setting. This allows us to use any concept from the log smooth setting.

5.2. Exploded $\mathbb{T}$ fibrations

We can now define exploded $\mathbb{T}$ fibrations analogously to exploded fibrations.

Definition 5.12. An exploded $\mathbb{T}$ fibration $\mathfrak{B}$ has the following structure.

(1) A stratified integral affine base B

(2) For each strata B^k_j, a fibration $\mathfrak{B}^k_j \longrightarrow B^k_j$ with log smooth $\mathbb{T}$ fibers for each $b \in B^k_j$,

$$M_b := (\mathbb{C}^*)^k \rtimes W_b^{n-2k}$$

These fibers are principal $(\mathbb{C}^*)^k$ bundles. They have an action of $T_b B^k_j$ given in integral coordinates by $(x_1, \dots, x_k)$ acts by multiplication by $(e^{x_1}, \dots, e^{x_k})$.

(3) A flat connection on each of these fibrations $\mathfrak{B}^k_j$ so that parallel transport gives log $\mathbb{T}$ smooth isomorphisms which are compatible with the action of $T_b B^k_j$ and affine parallel transport.

(4) For each inclusion ι in B, there is a dual log smooth $\mathbb{T}$ morphism,

$$\iota^\dagger : M_{\iota(b)} \longrightarrow M_b$$

so that

(a) For some strata $S \subset M_b$, $\iota^\dagger$ is a log smooth $\mathbb{T}$ isomorphism

$$\iota^\dagger : M_{\iota(b)} \longrightarrow \mathcal{N}_S M_b$$

(b) The induced action of $T_{\iota(b)}B$ on $\mathcal{N}_S M_b$ is compatible with $d\iota$. The action of any vector in $T_{\iota(b)}B$ pointing towards the interior of the strata moves $\mathcal{N}_S M_b$ in an outward direction.

(c) Every strata of M_b is the image of a unique isomorphism $\iota^\dagger$.

(d)

$$(\iota_1 \circ \iota_2)^\dagger = \iota_2^\dagger \circ \iota_1^\dagger$$

(e) These isomorphisms $\iota^\dagger$ are compatible with parallel transport within strata.

As with our definition of exploded fibrations, there is a global assumption that fibers have a globally defined action of $(\mathbb{C}^*)^k$ on them. If we patch together exploded $\mathbb{T}$ coordinate charts, then we will obtain a more sophisticated version of an exploded $\mathbb{T}$ fibration with an orbifold base and Seifert fibrations over strata.

Definition 5.13. A morphism of exploded $\mathbb{T}$ fibrations, $f : \mathfrak{B} \longrightarrow \mathfrak{C}$ is given by

(1) A stratified integral affine map of bases $f : B \longrightarrow C$.
(2) Log smooth $\mathbb{T}$ maps

$$f_b : M_b \longrightarrow M_{f(b)}$$

   (a) equivariant with respect to the action of T_bB and df
   (b) compatible with the isomorphisms $\iota^\dagger$ in the sense that

$$f_b \circ \iota^\dagger = F(\iota)^\dagger \circ f_{\iota(b)}$$

   (c) compatible with parallel transport within strata.

Note that any exploded $\mathbb{T}$ fibration is an exploded fibration and that any exploded $\mathbb{T}$ morphism is also a morphism of exploded fibrations.

Refinements are defined the same way as in the log smooth case. The results of section 4.2 hold for exploded $\mathbb{T}$ fibrations.

5.3. Almost complex structure

The tangent space $T\mathfrak{B}$ of an exploded fibration is an exploded fibration with the same base and fibers given by ${}^{\log}TM_b$, the log tangent space of fibers of $\mathfrak{B}$. This has a natural exploded structure because there is a natural identification of $\mathcal{N}_S{}^{\log}TM_b$ with ${}^{\log}T\mathcal{N}_SM_b$. The cotangent space $T^*\mathfrak{B}$ is a similarly well defined exploded fibration. A smooth section of one of these bundles is a section $\mathfrak{B} \longrightarrow T\mathfrak{B}$ which is a smooth exploded morphism. This allows us to define things like metrics or complex structures.

Definition 5.14. An almost complex structure on an exploded fibration $\mathfrak{B}$ is an endomorphism J of $T\mathfrak{B}$ which squares to minus the identity, given by a smooth section of $T\mathfrak{B} \otimes T^*\mathfrak{B}$.

(The tensor product above is over smooth functions on $\mathfrak{B}$ which are smooth morphisms to the exploded fibration which has a single point as its base and $\mathbb{R}$ as the fiber over that point.)

Definition 5.15. An (almost) complex exploded $\mathbb{T}$ fibration $\mathfrak{B}$ is an almost complex structure on $\mathfrak{B}$ considered as an exploded fibration which comes from giving each fiber M_b the structure of a (almost) complex $(\mathbb{C}^*)^k$ bundle over an (almost) complex manifold.

Note that this implies that substrata are holomorphic submanifolds.

There should be a good theory of holomorphic curves in almost complex exploded fibrations. In general, we need to consider holomorphic curves which are morphisms of a weaker type than described above. The theory of holomorphic curve in exploded $\mathbb{T}$ fibrations is better behaved. In this case, holomorphic curves are exploded $\mathbb{T}$ morphisms and when transversality conditions are met, the moduli space has an exploded $\mathbb{T}$ structure.

6. Examples of exploded $\mathbb{T}$ curves

6.1. Moduli space of stable exploded $\mathbb{T}$ curves

Definition 6.1. An exploded $\mathbb{T}$ curve is a complex exploded $\mathbb{T}$ fibration with a one dimensional base so that

1. The base is complete when given the metric that gives basic integral vectors unit size.
2. The fiber over any vertex is a compact Riemann surface with strata corresponding to a collection of marked points.

We will also call a holomorphic morphism of an exploded $\mathbb{T}$ curve to a almost complex exploded $\mathbb{T}$ fibration an exploded $\mathbb{T}$ curve.

Definition 6.2. An exploded $\mathbb{T}$ curve is stable if it has a finite number of automorphisms, and it is not a nontrivial refinement of another exploded $\mathbb{T}$ curve.

We now describe in detail the moduli space of stable exploded $\mathbb{T}$ curves (mapping to a point.)

Consider a Riemann surface Σ_c which is the fiber over the point c in the base C of an exploded $\mathbb{T}$ curve $\mathfrak{C}$. This has a number of marked points which correspond to boundary strata.

If $x \in \Sigma_c$ is such a marked point, then we can identify $\mathcal{N}_x\Sigma_c$ with $T_x\Sigma_c - 0$. The identification is given as follows: If x is locally defined by the vanishing of a boundary defining function z, a nonzero vector $v \in T_x\Sigma_c$ corresponds to $\nu \in \mathcal{N}_x\Sigma_c$ where $\nu(z) := dz(v)$. (This determines $\nu \in \mathcal{N}_x\Sigma_c$ by $\nu(z^k g) = (dz(v))^k g(x)$.)

Lemma 6.1. *A (stable) exploded $\mathbb{T}$ curve is equivalent to a (stable) punctured nodal Riemann surface with the following extra information at each node:*

1. *A length $l \in (0, \infty)$*
2. *A node consists of two marked points which are considered to be joined. There is a nondegenerate $\mathbb{C}$ bilinear pairing of the tangent spaces at these points.*

The length is the length of the edge joining two marked points. The $\mathbb{C}$ bilinear pairing comes from the identification of normal neighborhood bundles at these points.

Lemma 6.2. *Any finite order automorphism $\phi : \mathfrak{C} \longrightarrow \mathfrak{C}$ of a connected exploded $\mathbb{T}$ curve that is trivial on some fiber is the identity.*

Proof. Suppose that $\phi : \mathfrak{C} \longrightarrow \mathfrak{C}$ is such an automorphism. If ϕ_v is the identity for some vertex v, then the restriction of ϕ_v to the normal neighborhood bundle is the identity, so ϕ_e is also the identity for e any point on an edge connected to v (and ϕ is the identity on those edges.)

If ϕ_e is the identity for e some point in an edge, ϕ must be the identity on the edge, and all fibers on the edge. Suppose that v is a vertex on the end of such an edge. $\phi_v : \Sigma_v \longrightarrow \Sigma_v$ must be a finite order automorphism of Σ_v that fixes the marked point x attached to the edge, and that fixes the tangent space $T_x\Sigma_v$. The only such automorphism is the identity. (This can be seen locally, if $\phi_v(z) = z + cz^n + O(z^{n+1})$, then the k-fold composition of ϕ_v must be $z + kcz^n + O(z^{n+1})$, so if ϕ_v is not the identity, then it is not of finite order.)

□

In particular, the above lemma tells us that the automorphisms of nodal Riemann surfaces that are unbalanced in the sense that they twist one side of a node more than the other are not automorphims of exploded $\mathbb{T}$ curves. An automorphim such as this lifts to an isomorphism of an exploded curve with an exploded curve obtained by modifying the bilinear pairings at nodes. Isomorphims such as this will act like automorphims when we describe the orbifold structure of moduli spaces of exploded curves.

Definition 6.3. A lifted automorphism of an exploded $\mathbb{T}$ curve is an isomorphism of the exploded $\mathbb{T}$ curve to an exploded $\mathbb{T}$ curve obtained by modifying bilinear pairings at nodes.

The lifted automorphisms of an exploded $\mathbb{T}$ curve are in one to one correspondence with a subgroup of the automorphisms of the underlying nodal curve. In the case that all edges have equal length, the lifted automorphisms correspond to the automorphisms of the underlying nodal curve.

The moduli space $\mathfrak{M}_{g,n}$ of stable exploded $\mathbb{T}$ curves with genus g and n marked points has the structure of an orbifold exploded $\mathbb{T}$ fibration. An orbifold exploded fibration has uniformising charts $(\mathfrak{U}, G)$ where $\mathfrak{U}$ is an exploded $\mathbb{T}$ coordinate chart, and G is a group acting on $\mathfrak{U}$.

Uniformising charts for $\mathfrak{M}_{g,n}$ are constructed as follows: Choose holomorphic uniformising charts (U, G) for Deligne Mumford space $\bar{\mathcal{M}}_{g,n}$ so that the boundary strata contained in U consist of the vanishing of $\{z_1, \dots, z_k\}$, the first k coordinates of U. This makes U a log smooth $\mathbb{T}$ coordinate chart where $\{z_1, \dots, z_k\}$ are boundary defining functions. These correspond to k nodes in the curve over $z_1 = \dots = z_k = 0$. Note that the action of G on U and any transitions between coordinate charts are log $\mathbb{T}$ smooth.

Uniformising charts for $\mathfrak{M}_{g,n}$ are then given by the explosion $(\text{Expl}(U), G)$. The base of $\text{Expl}(U)$ has coordinates $(l_1, \dots, l_k) \in [0, \infty)^k$. The fiber over the strata where $l_i = 0$ for $i \notin I$ is given by $\mathcal{N}_{S_I}U$ where S_I is the strata given by $z_i = 0$ for $i \in I$. The action of G and transition maps are given by noting that any log smooth $\mathbb{T}$ map $U_1 \longrightarrow U_2$ gives a natural exploded $\mathbb{T}$ morphism $\text{Expl}(U_1) \longrightarrow \text{Expl}(U_2)$.

We must now see why points in $\mathfrak{M}_{g,n}$ (which we will consider to be the image of a morphism of a point to $\mathfrak{M}_{g,n}$) correspond to exploded $\mathbb{T}$ curves. We do this for each uniformising chart $\text{Expl}(U)$. A point in the strata where $l_i = 0$ for $i \notin I$ has the information of a nodal curve corresponding to $s \in S_I$ plus an evaluation $\nu_s \in \mathcal{N}_s U$ and lengths l_i for $i \in I$. These lengths correspond to the length associated to each node. Note that if this point is the image of a morphism, then $l_i > 0$ for $i \in I$, and these are the only nodes that our curve has. To describe this as a moduli space of exploded curves, we must also say how the bilinear pairing of tangent spaces at a node is given by the above choice of an evaluation $\nu_s \in \mathcal{N}_s U$.

Recall that there is a universal curve $\bar{\mathcal{M}}_{g,n+1} \longrightarrow \bar{\mathcal{M}}_{g,n}$ given by forgetting the point labeled with $(n+1)$. This is log $\mathbb{T}$ smooth as it is holomorphic and the inverse image of boundary strata are boundary strata, so we also have a map of normal neighborhood bundles $\mathcal{N}\bar{\mathcal{M}}_{g,n+1} \longrightarrow \mathcal{N}\bar{\mathcal{M}}_{g,n}$. If the forgotten point is near the first node, this map can be given in coordinates locally near $z_1 = 0$ by

$$(z_1^+, z_1^-, z_2, \dots) \mapsto (z_1^+ z_1^-, z_2, \dots)$$

(where dots are completed with the identity). The curve over the point $z_1 = 0$ gives local coordinates for our node, considered as two disks with coordinates z_1^+ and z_1^- joined over the points $z_1^+ = 0$ and $z_1^- = 0$. We have a map of the normal neighborhood bundle of the strata where $z_1^+ = z_1^- = 0$ to the normal neighborhood bundle of the strata $z_1 = 0$, given in our coordinates by the above map, where z_1^+ and z_1^- now give coordinates for the tangent space to our curve at the node. The bilinear pairing in these coordinates is

$$(z_1^+, z_1^-) \mapsto \frac{z_1^+ z_1^-}{\nu_s(z_1)}$$

This is independent of coordinate choices, and the choice of pairings for each of the k nodes is equivalent to the choice of ν_s.

A second way to see the relationship between pairings and ν_s is to view the pairings as gluing information. In particular, given the curve over s and choice of coordinates z_i^+, z_i^- around its nodes, we can glue these neighborhoods with the help of a small complex parameter c_i, identifying

$$z_i^+ = \frac{c_i}{z_i^-}$$

This defines a gluing map $\mathbb{C}^k \longrightarrow U$ defined near $c_i = 0$. Giving $\mathbb{C}^k$ the strata $c_i = 0$, this map is log $\mathbb{T}$ smooth, and the restriction $\mathcal{N}_0\mathbb{C}^k \longrightarrow \mathcal{N}_s U$ is an isomorphism. If the pullback of ν_s to $\mathcal{N}_0\mathbb{C}^k$ has coordinates $(\nu_{s,1}, \dots, \nu_{s,k})$, then the pairing between z_i^+ and z_i^- is given by

$$(z_i^+, z_i^-) = \frac{z_i^+ z_i^-}{\nu_{s,i}}$$

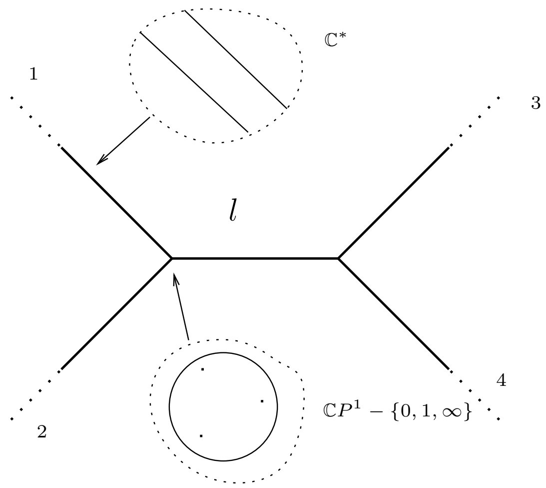
The above discussion can be summarized in the following lemma.

Lemma 6.3. *The forgetful map*

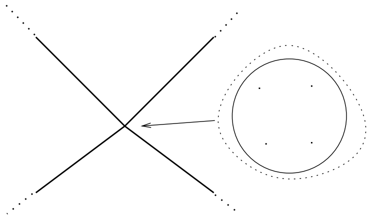
$$\mathfrak{M}_{g,n+1} \longrightarrow \mathfrak{M}_{g,n}$$

is log $\mathbb{T}$ smooth. Given any point $p \longrightarrow \mathfrak{M}_{g,n}$, the fiber over p is the exploded curve corresponding to p quotiented out by its automorphisms.

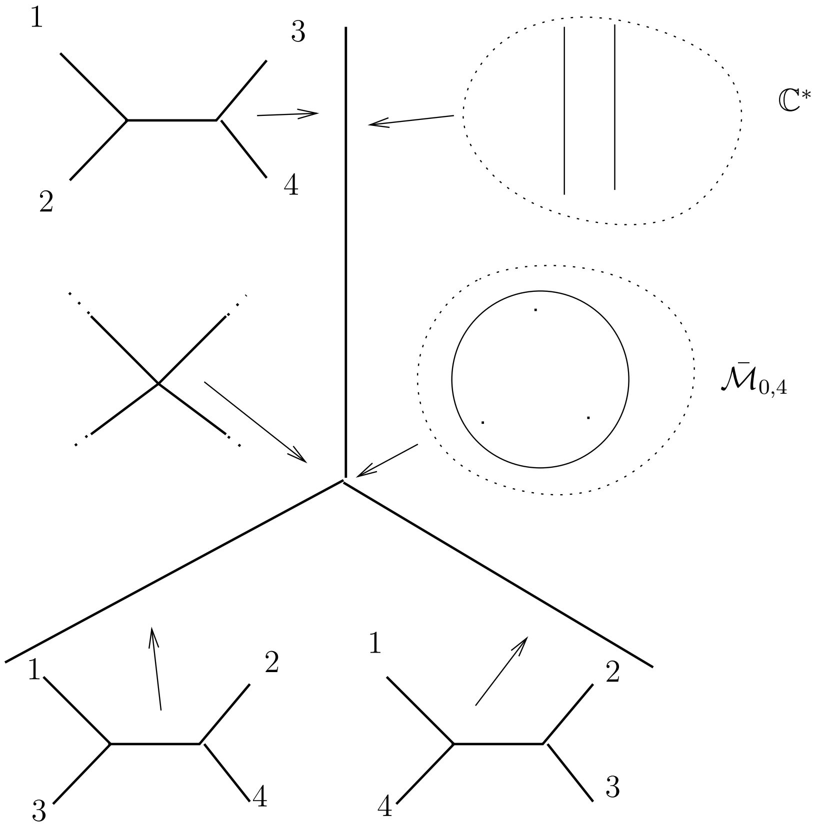
For example, $\mathfrak{M}_{0,3}$ is a point, and $\mathfrak{M}_{0,4}$ is equal to an exploded $\mathbb{T}$ curve with 3 punctures. A curve in $\mathfrak{M}_{0,4}$ might look like the following curve or a similar one where the punctures 1 and 3 or 1 and 4 are grouped together.



For each picture of the above type, there is a $\mathbb{C}^*$ family of curves in $\mathfrak{M}_{0,4}$ corresponding to different identifications of normal neighborhood bundles along the interior edge. The length l of this edge gives a coordinate for a strata of $M_{0,4}$. The above observation tells us that the fiber over each point in this strata is $\mathbb{C}^*$. The different pairings of punctures give 3 such strata. the end $l = 0$ of this strata is glued to a strata with curves looking like the following:



The curves with a picture such as this form a family parametrized by the complex structure of the fiber over the central vertex. This gives a strata of $\mathfrak{M}_{0,4}$ consisting of a point with fiber equal to $\bar{\mathcal{M}}_{0,4}$ with 3 boundary strata. A picture of $\mathfrak{M}_{0,4}$ is as follows



The identification between the normal neighborhood bundles of boundary strata of $\bar{\mathcal{M}}_{0,4}$ and the gluing data $\mathbb{C}^*$ is explained above.

6.2. Exploded $\mathbb{T}$ curves in smooth manifolds

A connected smooth manifold M can be regarded as an exploded $\mathbb{T}$ fibration with base consisting of a point, and fiber over that point being M. If M is given an almost complex structure, we can then consider holomorphic exploded $\mathbb{T}$ curves in M. A holomorphic exploded $\mathbb{T}$ curve in this setting has the information of a nodal holomorphic curve along with a positive length and a $\mathbb{C}$ bilinear pairing of tangent spaces assigned to each node. This extra information corresponds to exploding the moduli space instead of compactifying it. The exploded $\mathbb{T}$ curve is stable if the underlying nodal holomorphic curve is stable.

6.3. Symplectic sum

Given symplectic manifolds M_1^{2n} and M_2^{2n} along with symplectic submanifolds,

$$N^{2(n-1)} \hookrightarrow M_1^{2n} \text{ and } N^{2(n-1)} \hookrightarrow M_2^{2n}$$

so that the normal bundle of N in M_1 is dual to the normal bundle of N in M_2, we can construct the symplectic sum of M_1 and M_2 over N, $M_1 \#_N M_2$. The question of how to find holomorphic curves in $M_1 \#_N M_2$ in terms of holomorphic curves in M_1, M_2 and the normal bundle to N has been answered in [4]. This involves a degeneration of complex structure which can be viewed as giving the following exploded $\mathbb{T}$ fibration.

First choose an almost complex structure on M_1 and M_2 so that N is an almost complex submanifold, and the normal bundle to N in M_1 is a complex bundle isomorphic to the dual of the normal bundle in M_2. M_1 and M_2 are log smooth $\mathbb{T}$ spaces with a single substrata N. Note that the normal neighborhood bundle $\mathcal{N}_N M_1$ is equal to the normal bundle of N minus the zero section.

The exploded fibration has a base equal to the interval $[1, 2]$ with fiber over the strata 1 being M_1, fiber over the strata 2 being M_2. The fiber over any point in the strata $[1, 2]$ is equal to $\mathcal{N}_N M_2 = \mathbb{C}^* \rtimes N$. This is identified with $\mathcal{N}_N M_1$ using the identification of the normal bundle of N in M_1 with the dual of the normal bundle of N in M_2. Multiplication by a vector c in the tangent space to $[1, 2]$ corresponds to multiplication by e^c if the fiber is viewed as the normal bundle of N in M_2, and to multiplication by e^{-c} when viewed as the normal bundle of N in M_1.

6.4. Relationship with tropical geometry

An easy example of an exploded $\mathbb{T}$ fibration is $\mathfrak{T}\mathbb{R}^n$, which has a base given by $\mathbb{R}^n$ and fibers given by $(\mathbb{C}^*)^n$. We will now examine holomorphic exploded $\mathbb{T}$ curves

$$f : \mathfrak{C} \longrightarrow \mathfrak{T}\mathbb{R}^n$$

At any vertex c in the base C of $\mathfrak{C}$, we have a holomorphic map defined on the interior of the fiber Σ_c.

$$f_c : \Sigma_c \longrightarrow (\mathbb{C}^*)^n$$

The fact that this is a log smooth $\mathbb{T}$ morphism implies that this must extend to a meromorphic map over punctures. In particular, in local coordinates z around a puncture,

$$f_c(z) := (z^{\alpha_1} g_1(z), \dots, z^{\alpha_n} g_n(z))$$

where $\alpha_i \in \mathbb{Z}$ and g_i are holomorphic $\mathbb{C}^*$ valued functions which extend over punctures. The restriction of f_c to the normal neighborhood is then given by

$$f_c(z) := (z^{\alpha_1} g_1(0), \dots, z^{\alpha_n} g_n(0))$$

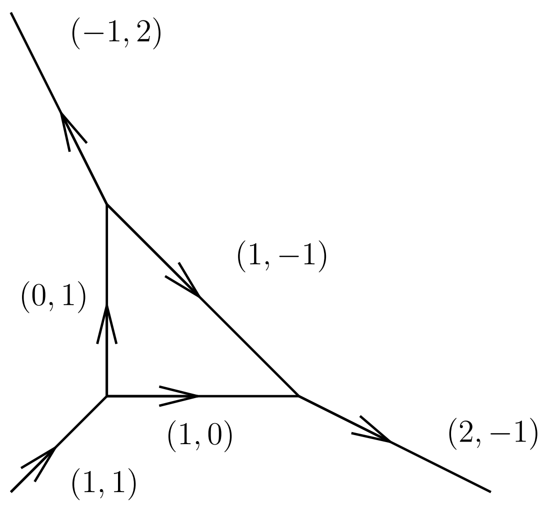
(The coordinate z in this case is the restriction of the above coordinate to the normal neighborhood bundle.)

This then defines the fiber maps on the edge attached to this puncture, so if e is a point on this edge, then

$$f_e(z) := (z^{\alpha_1} g_1(0), \dots, z^{\alpha_n} g_n(0))$$

This has implications for the map $f : C \longrightarrow \mathbb{R}^n$ of bases. It means that this edge viewed as oriented away from the vertex c must travel in the integral direction $\vec{\alpha} := (\alpha_1, \dots, \alpha_n)$. If the edge has length l, then the image of the edge is $l\vec{\alpha}$.

df restricted to this edge gives a map $\mathbb{Z} \longrightarrow \mathbb{Z}^n$ given by $k \mapsto k\vec{\alpha}$. Call this the momentum of the edge, and $\vec{\alpha}$ the momentum of the edge exiting the vertex. The sum of all momentums exiting a vertex is zero. This is because number of poles of a meromorphic function is equal to the number of zeros. It can be viewed as a balancing or conservation of momentum condition on the base of a holomorphic curve.



The special case when the base of an exploded curve is a trivalent graph and the fiber over all vertices is equal to a three punctured sphere is interesting, because most of the information in the exploded curve can be read off from the base. In this case, the map of bases $f : C \longrightarrow \mathbb{R}^n$ is called a tropical curve. The technique of counting tropical curves to get holomorphic curve invariants has been used quite successfully (for example, see [5]).

Definition 6.4. A tropical exploded $\mathbb{T}$ fibration is an exploded $\mathbb{T}$ fibration in which every fiber is some product $(\mathbb{C}^*)^k \times \mathbb{C}P^{n-k}$ where $\mathbb{C}P^{n-k}$ is given the log smooth $\mathbb{T}$ structure with boundary strata given by $(n-k+2)$ generic hyperplanes.

The base of a tropical exploded $\mathbb{T}$ fibration is a tropical space. Holomorphic curves in these tropical exploded $\mathbb{T}$ fibrations with tropical domains have bases which are tropical curves.

7. Fiber product

The category of exploded fibrations has a good intersection theory.

Definition 7.1. Given two exploded morphisms with the same target,

$$\mathfrak{A} \xrightarrow{f} \mathfrak{C} \xleftarrow{g} \mathfrak{B}$$

f and g are transverse if all the following maps are transverse restricted to the interior of their domains

$$M_a \xrightarrow{f_a} M_c \xleftarrow{f_b} M_b \text{ when } f(a) = c = f(b)$$

The fiber product of transverse exploded morphisms is well defined, however it is sometimes not an exploded fibration of the type we have described up until now. It may have strata which are locally modeled on open subsets of

$$\{b : \alpha_j \cdot b \geq 0\} \subset \mathbb{R}^n$$

where $\{\alpha_j\}$ is some set of integer vectors. There is a correspondingly more permissive definition of log smooth coordinate charts for fibers.

Theorem 7.1. *Suppose that f and g are transverse exploded morphisms,*

$$\mathfrak{A} \xrightarrow{f} \mathfrak{C} \xleftarrow{g} \mathfrak{B}$$

so that for any pair of strata A^k_j and B^l_i in the base of $\mathfrak{A}$ and $\mathfrak{B}$ so that $M_a\ {}_{f_a}\times_{g_b} M_b$ is nonempty for some $a \in A$ and $b \in B$, the subset $\{(a,b) : f(a) = g(b)\} \subset A \times B$ is an integral affine space locally modeled on open subsets of $[0,\infty)^n$.

Then there exists an exploded fibration called their fiber product

$$\mathfrak{A}\ {}_f\times_g \mathfrak{B}$$

with maps to $\mathfrak{A}$ and $\mathfrak{B}$ so that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{A}\ {}_f\times_g \mathfrak{B} & \longrightarrow & \mathfrak{A} \\ \downarrow & & \downarrow \\ \mathfrak{B} & \longrightarrow & \mathfrak{C} \end{array}$$

and with the usual universal property that given any commutative diagram

$$\begin{array}{ccc} \mathfrak{D} & \longrightarrow & \mathfrak{A} \\ \downarrow & & \downarrow \\ \mathfrak{B} & \longrightarrow & \mathfrak{C} \end{array}$$

there exists a unique morphism $\mathfrak{D} \longrightarrow \mathfrak{A} \ {}_f\times_g \mathfrak{B}$ *so that the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{D} & \rightarrow & \mathfrak{A} \\ \downarrow & \searrow & \uparrow \\ \mathfrak{B} & \leftarrow & \mathfrak{A} \ {}_f\times_g \mathfrak{B} \end{array}$$

Proof. First, we check the theorem in the case that g is a map of a point into $\mathfrak{C}$. In this case, the image of g is some point $c \in C$ and a point $x \in M_c$ in the fiber over c. As x is the image of a morphism, it must be in the interior of M_c. We are interested in $f^{-1}(x)$.

First consider $f^{-1}(c)$. The base of $f^{-1}(x)$ will be made up of copies of strata of $f^{-1}(c)$, with each strata of $f^{-1}(c)$ appearing a number of times equal to the number of connected components of $f^{-1}(x)$ over it. Call a strata of $f^{-1}(c)$ empty if $f^{-1}(x)$ is empty over the strata. We will discard such strata.

Note that for any $a \in A$, f_a is transverse to x on the interior of M_a. This property is preserved if we take refinements of $\mathfrak{A}$. We would like f_a restricted to the interior of any strata of M_a to be transverse to x. To achieve this, consider a local refinement of $\mathfrak{A}$ coming from a subdivision of A close to a so that the non empty strata of $f^{-1}(c)$ are strata in the subdivision. (There may be no global subdivision satisfying this, but it is easy to construct one locally). Note that any morphism $\mathfrak{D} \longrightarrow \mathfrak{A}$ contained in $f^{-1}(x)$ lifts locally to a unique morphism to the local refinement of $\mathfrak{A}$.

After taking such a refinement, if $a \in f^{-1}(c)$ is in a non empty strata, then f_a is constant in any affine direction. Also f_a restricted to the interior of any strata $S \subset M_a$ is transverse to x. This is because f_a restricted to $\mathcal{N}_S M$ must be transverse to x, but if the interior of $f_a(S)$ intersects x, it must be contained in the interior of M_c, and in such a case f_a restricted to $\mathcal{N}_S M_a$ is just the pullback of f_a restricted to S. This makes $f_a^{-1}(x)$ some number of log smooth spaces. Affine coordinates are just the restriction of affine coordinates on M_a, and boundary defining functions are the restriction of boundary defining functions for strata that $f_a^{-1}(x)$ intersects. A single boundary of M_a may correspond to several boundaries in $f_a^{-1}(x)$. Note that $f_a^{-1}(x)$ does not depend on the particular choice of local refinement.

If $\iota : a \mapsto \iota(a)$ is an inclusion in $f^{-1}(c)$ and S is the strata in M_a corresponding to ι, then

$$f_{\iota(a)}^{-1}(x) = \left(f_a|_{\mathcal{N}_S M_a}\right)^{-1}(x)$$

If we take the non empty strata of $f^{-1}(c)$ to be the base of $f^{-1}(x)$, $f_a^{-1}(x)$ to be the fibers, and restrict parallel transport and isomorphisms $\iota^\dagger$ from (the local refinements of) $\mathfrak{A}$, then $f^{-1}(x)$ satisfies all the axioms of an exploded fibration apart from having connected fibers. To remedy this, have a copy of each strata of $f^{-1}(c)$ for each fiber component, and connect them by inclusions if one fiber is the normal neighborhood of some strata in the other. This exploded fibration has the required universal property.

Now, we consider the general case of the transverse intersection of two morphisms, f and g.

Cover the target $\mathfrak{C}$ with exploded coordinate charts.

$$\begin{array}{l} \mathfrak{U}_\alpha \longrightarrow \mathfrak{C} \\ \downarrow \\ \mathfrak{R}^n \end{array}$$

Note that $f^{-1}(\mathfrak{U}_\alpha)$ is an exploded fibration, so we can restrict f to $f^{-1}(\mathfrak{U}_\alpha)$.

On these charts, we can consider

$$f - g : f^{-1}(\mathfrak{U}_\alpha) \times g^{-1}(\mathfrak{U}_\alpha) \longrightarrow \mathfrak{R}^n$$

Recall that $\mathfrak{R}^n$ is the exploded fibration with base equal to $\mathbb{R}^n$ and fibers all equal to $\mathbb{R}^n$. As f is transverse to g, $(f-g)$ is transverse to the point 0 in the fiber over 0, and $(f-g)^{-1}(0)$ is an exploded fibration. This constructs $f^{-1}(\mathfrak{U}_\alpha)\ {}_f\!\times_g g^{-1}(\mathfrak{U}_\alpha)$. This has the correct universal property because any commutative diagram

$$\begin{array}{ccc} \mathfrak{D} & \longrightarrow & f^{-1}(\mathfrak{U}_\alpha) \\ \downarrow & & \downarrow \\ g^{-1}(\mathfrak{U}_\alpha) & \longrightarrow & \mathfrak{U}_\alpha \end{array}$$

gives a morphism $\mathfrak{D} \longrightarrow f^{-1}(\mathfrak{U}_\alpha) \times g^{-1}(\mathfrak{U}_\alpha)$ contained in the inverse image of 0.

Doing the same for all coordinate charts and patching together the result define an exploded fibration $\mathfrak{A}\ {}_f\!\times_g \mathfrak{B}$ with the required universal property.

□

Note that if f and g are transverse exploded $\mathbb{T}$ morphisms satisfying the above condition on the base, $\mathfrak{A}\ {}_f\!\times_g \mathfrak{B}$ is an exploded $\mathbb{T}$ fibration.

An important example is given by the forgetfull morphisms

$$\pi : \mathfrak{M}_{g,n+1} \longrightarrow \mathfrak{M}_{g,n}$$

This map is a surjective submersion restricted to the interior of each fiber, so any smooth exploded morphism $f : \mathfrak{F} \longrightarrow \mathfrak{M}_{g,n}$ must be transverse to π.

Definition 7.2. An exploded fibration is complete if any metric on it is complete, and the base has a finite number of strata, all of which are complete with respect to the integral affine connection. (A metric on an exploded fibration $\mathfrak{B}$ is a symmetric, positive definite, exploded section of $T^*\mathfrak{B} \otimes T^*\mathfrak{B}$.)

Definition 7.3. An exploded fibration is compact if it is complete and the base is compact.

Lemma 7.2. *If $\mathfrak{A}$ and $\mathfrak{B}$ are complete and $f : \mathfrak{A} \longrightarrow \mathfrak{C}$ and $g : \mathfrak{B} \longrightarrow \mathfrak{C}$ are transverse, then $\mathfrak{A}\ {}_f\times_g \mathfrak{B}$ is complete. It $\mathfrak{A}$ and $\mathfrak{B}$ are compact, then $\mathfrak{A}\ {}_f\times_g \mathfrak{B}$ is compact.*

We now want to say that two exploded morphisms to the same target are generically transverse. In the smooth case, statements of the following sort can be useful if $f : M \longrightarrow N$ is transverse to g on the boundary of M and g is proper, then there is a small isotopy of f fixed on the boundary to a map transverse to g. In this exploded setup, instead of a boundary, we will have a subset of strata which are closed under inclusion. This means that if B^k_j is inside this subset, then the target of any inclusion of B^k_j must be contained in this subset. For isotopies, instead of the unit interval $[0, 1]$, we will have the explosion of the unit interval, $\mathrm{Expl}[0, 1]$.

Lemma 7.3. *Given two smooth exploded morphisms,*

$$\mathfrak{A} \xrightarrow{f} \mathfrak{C} \xleftarrow{g} \mathfrak{B}$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are complete, and the strata of A are simply connected, then there exists a small isotopy

$$F : Expl[0, 1] \times \mathfrak{A} \longrightarrow \mathfrak{C}$$

so that

1. *the restriction to any fiber of $\mathfrak{A}$, $F_a : [0, 1] \times M_a \longrightarrow M_{F(a)}$ is a C^∞ small isotopy.*
2. *$F(t, \cdot)$ is independent of t in a neighborhood of $t = 0$ and $t = 1$, equal to f when t is small, and transverse to g when t is close to 1.*

Moreover, if f is transverse to g on a subset of strata closed under inclusions, then the isotopy can be chosen to be fixed on those strata.

Proof. We construct the isotopy first over fibers that have no boundary strata and then extend it over fibers with strata of higher codimension.

If M_a has no boundary strata, it is some affine $\mathbb{R}^n$ bundle over a compact manifold. As the isotopy can't change df_a in affine directions, the isotopy is determined by its restriction to some section of this bundle. Note that to achieve transversality, we can quotient the target of f_a by the directions spanned by df of affine directions. The task is then to find an isotopy so that the image of this section is transverse to the image of g. As $\mathfrak{B}$ is complete, the image of g in this fiber consists of the image of a finite number of smooth proper maps of manifolds with boundary and corners, so a generic smooth map is transverse to g and there exists a C^∞ small isotopy of f_a making it transverse to g as required. (We chose the trivial isotopy if f_a is already transverse.)

This isotopy can be extended compatibly via parallel transport to all fibers of $\mathfrak{A}$ that have no boundary (this uses that the strata of A are simply connected, so there is no monodromy). Now suppose that the isotopy has been defined compatibly on all fibers that only have strata of codimension less than k. We now must extend it to an isotopy on fibers with strata of maximum codimension k.

Suppose that M_a is such a fiber. As any strata S of nonzero codimension has substrata of dimension at most $k - 1$, we have defined the isotopy already on $\mathcal{N}_S M_a$. Viewing our isotopy as given by exponentiating some smooth, time dependant vector field, we just need to extend this vector field over M_a. It is defined on the boundary of M_a so that it is continuous, and smooth restricted to any boundary or corner strata. As such it extends to a smooth vector field on the interior of M_a, which can be chosen C^∞ small depending on its C^∞ size on the boundary strata. (One can see locally why this is true: A continuous function defined the boundary of $[0,\infty)^n$ which is smooth on boundary and corner strata extends smoothly to the interior as follows. Denote by π_S the projection to the strata S, and let $\sigma(S)$ be 1 if S has odd codimension, and -1 if S has even codimension. Then we can set $f := \sum \sigma(S) f \circ \pi_S$.)

We now have a C^∞ small isotopy of f_a that restricts to any nontrivial normal neighborhood bundle to be our previously defined isotopy. As this restriction to the normal neighborhood bundle is transverse to the image of g near $t = 1$, and the space of maps transverse to the image of g is open, this isotopy is transverse to the image of g near the boundary and $t = 1$. We can then further modify this isotopy on the interior of M_a to be transverse to g everywhere near $t = 1$. This isotopy can be chosen C^∞ small, and if f_a is already transverse to g and we have chosen the isotopy trivial on the normal neighborhood bundles of boundary strata of M_a, the this isotopy can be trivial.

Thus we can extend the isotopy as required.

□

A very similar statement holds for exploded $\mathbb{T}$ fibrations, however there is one small complication: $\text{Expl}[0,1] \times \mathfrak{A}$ is not an exploded $\mathbb{T}$ fibration. We can define a category of mixed exploded fibrations which includes fiber products of exploded torus fibrations and exploded fibrations. The fibers of these are locally modeled on $\{z^{\alpha_i} \in [0,\infty)\} \subset \mathbb{C}^n \times \mathbb{R}^k$ where $\{\alpha_i\}$ are multiindices in $\mathbb{Z}^n$ which form a basis for some subspace of $\mathbb{Z}^n$, and log smooth functions are the restrictions of functions of the form $z^\alpha g$ where g is a smooth $\mathbb{C}^*$ valued function. The case when the set $\{\alpha_i\}$ spans $\mathbb{Z}^n$ corresponds to exploded fibrations, and the case where this set is empty corresponds to exploded torus fibrations. Definitions of exploded fibrations using this mixed version of log smooth spaces are the same as our earlier definitions. The results of section 4.2 and this section apply for this more general category.

8. Moduli space of smooth exploded $\mathbb{T}$ curves

In order to describe the structure on the moduli space of smooth exploded curves, we will need to use notions from the theory of polyfolds developed by Hofer, Wysocki and Zehnder in [6].

Definition 8.1. A Frechet splicing core K is the fixed point set of a smooth family of projections

$$\pi : F \times [0,\infty)^k \to F \times [0,\infty)^k$$

where $\pi(\cdot, x)$ gives a projection on the Frechet space $F \longrightarrow F$ and $\pi(\cdot, 0)$ is the identity.

A log smooth function $f \in {}^{\log}C^\infty\left(F\times[0,\infty)^k\right)$ is a function defined on the interior of $F\times[0,\infty)^k$ so that

$$f=\sum \alpha_i \log x_i + g$$

where $\alpha_i\in\mathbb{Z}$ and g is a smooth function on $F\times[0,\infty)^k$.

Definition 8.2. A log smooth Frechet polyfold is a Hausdorff second countable topological space M along with a sheaf of log smooth functions ${}^{\log}C^\infty(M)$ so that around each point $p\in M$, there exists an open set U_p and a homeomorphism onto some Frechet splicing core $\phi_p : U_p \longrightarrow K\subset F_p\times[0,1)^{k_p}$ sending p to the fiber over 0 so that

$${}^{\log}C^\infty(U_p)=\phi_p^*\left({}^{\log}C^\infty(F_p\times[0,\infty)^{k_p})\right)$$

p is said to be in a strata of codimension k_p.

Note that if M is finite dimensional it is simply a log smooth space, as then splicing cores must be diffeomorphic to $F\times[0,\infty)^k$.

The subset of M consisting of points with codimension k is a smooth Frechet manifold. For each connected component of these points of codimension k, there exists a unique log smooth Frechet polyfold S and a continuous map $S\longrightarrow M$ so that the interior of S maps diffeomorphically to this component. Call all such S the strata of M.

We can define affine $\mathbb{R}^n$ bundles over log smooth Frechet polyfolds, log smooth maps, normal neighborhood bundles and log smooth morphisms exactly as in the finite dimensional case. With this done, the definition of a Frechet exploded fibration is identical to the definition of an exploded fibration where fibers are log smooth Frechet polyfolds.

We can similarly define Frechet exploded $\mathbb{T}$ fibrations.

Definition 8.3. Given a closed 2 form ω on the exploded $\mathbb{T}$ fibration $\mathfrak{B}$, the smooth exploded $\mathbb{T}$ curve $f:\mathfrak{C}\longrightarrow\mathfrak{B}$ is ω-stable if C has a finite number of strata and the integral of $f^*\omega$ is positive on any unstable fiber Σ_v over a vertex of C.

What follows is a rough sketch of the perturbation theory of ω-stable holomorphic curves.

Conjecture 8.1. *The moduli space $\mathfrak{M}^{sm}_{g,n}(\mathfrak{B},\omega)$ of ω-stable smooth exploded curves in $\mathfrak{B}$ with genus g and n marked points has a natural structure of a Frechet orbifold exploded fibration.*

If $\mathfrak{B}$ has an almost complex structure J, we can take $\bar\partial$ of a smooth exploded $\mathbb{T}$ curve $f:\mathfrak{C}\longrightarrow\mathfrak{B}$. If j denotes the complex structure on $\mathfrak{C}$, this is defined as

$$\bar\partial f := \frac{1}{2}(df + J\circ df\circ j)$$

This takes values in antiholomorphic sections of $T^*\mathfrak{C}\otimes f^*(T\mathfrak{B})$ which vanish on the fibers over the edges of $\mathfrak{C}$.

Conjecture 8.2. *The space of these antiholomorphic sections has the structure of a Frechet orbifold exploded fibration* $\mathfrak{E}_{g,n}(\mathfrak{B},\omega)$*, which is a vector bundle over* $\mathfrak{M}^{sm}_{g,n}(\mathfrak{B},\omega)$*. The* $\bar\partial$ *equation defines a smooth section of this vector bundle.*

$$\begin{array}{l}\mathfrak{E}_{g,n}(\mathfrak{B},\omega)\\ \downarrow\uparrow \bar\partial\\ \mathfrak{M}^{sm}_{g,n}(\mathfrak{B},\omega)\end{array}$$

Conjecture 8.3. *The section* $\bar\partial$ *is 'Fredholm' in the following sense: There is a cover of* $\mathfrak{M}_{g,n}(\mathfrak{B},\omega)$ *by exploded coordinate neighborhoods* $(\mathfrak{U},G)$ *covered by coordinates* $(\mathfrak{E},G)$ *on* $\mathfrak{E}_{g,n}(\mathfrak{B},\omega)$ *so that there exists the following*

(1) *An identification of* $\mathfrak{U}$ *with a coordinate neighborhood in a vector bundle over a finite dimensional exploded* $\mathbb{T}$ *fibration* $\mathfrak{F}$

$$\begin{array}{rl}\mathfrak{U}\xrightarrow{\psi} & \mathfrak{V}\\ & \downarrow\\ & \mathfrak{F}\end{array}$$

(2) *An identification of* $\mathfrak{E}$ *with a coordinate neighborhood in a fibration*

$$\begin{array}{c}\mathfrak{E}\subset\mathfrak{V}\oplus\tilde{\mathfrak{E}}\\ \downarrow\\ \mathfrak{F}\end{array}$$

so that the projection $\mathfrak{E}\to\mathfrak{U}$ *is given by* $(v,e)\mapsto v$*, and the zero section* $\mathfrak{U}\longrightarrow\mathfrak{E}$ *is given by* $v\mapsto(v,0)$*.*

(3) *An exact sequence of vector bundles over* $\mathfrak{F}$

$$0\longrightarrow\mathbb{R}^k\times\mathfrak{F}\longrightarrow\tilde{\mathfrak{E}}\xrightarrow{\pi_{\mathfrak{V}}}\mathfrak{V}\longrightarrow 0$$

so that the graph of $\pi_{\mathfrak{V}}\circ\bar\partial$ *is the diagonal in* $\mathfrak{V}\oplus\mathfrak{V}$

In the non algebraic case, the regularity of maps and identifications in the above conjecture may be questionable.

The above conjecture says roughly that the intersection theory of $\bar\partial$ with the zero section can be modelled locally on the following finite dimensional problem: first, the solutions of the equation $\pi_{\mathfrak{V}}\circ\bar\partial$ can be identified with $\mathfrak{F}$. $\bar\partial$ of such a solution then has values in the $\mathbb{R}^k$ from the above exact sequence, so we get a map $\bar\partial:\mathfrak{F}\longrightarrow\mathbb{R}^k$. We are interested in the transverse intersection of this map with 0.

We would like to take the transverse intersection of $\bar\partial$ with the zero section and obtain the moduli space of holomorphic curves as a smooth exploded $\mathbb{T}$ fibration. The problem with this is that as we are dealing with orbifolds, we can't necessarily perturb the zero section to get it transverse to the zero section. Following [7], we can take a resolution of the zero section by a nonsingular weighted branched exploded fibration $\tilde{\mathfrak{M}} \to \mathfrak{M}^{sm}_{g,n}(\mathfrak{B},\omega)$, and then perturb that to be transverse to $\bar{\partial}$, obtaining a representation of the virtual moduli space of holomorphic curves inside $\mathfrak{M}^{sm}_{g,n}(\mathfrak{B},\omega)$ as a smooth weighted branched exploded $\mathbb{T}$ fibration.

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Asymptotically maximal real algebraic hypersurfaces of projective space

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Abstract. Using the combinatorial patchworking, we construct an asymptotically maximal (in the sense of the generalized Harnack inequality) family of real algebraic hypersurfaces in an n-dimensional real projective space. This construction leads to a combinatorial asymptotic description of the Hodge numbers of algebraic hypersurfaces in the complex projective spaces and to asymptotically sharp upper bounds for the individual Betti numbers of primitive T-hypersurfaces in terms of Hodge numbers of the complexifications of these hypersurfaces.

1. Introduction

In 1876 A. Harnack published a paper [Har76] where he found an exact upper bound for the number of connected components for a curve of a given degree. Harnack proved that the number of components of a real plane projective curve of degree m is at most $\frac{(m-1)(m-2)}{2}+1$. On the other hand, for each natural number m he constructed a non-singular real projective curve of degree m with $\frac{(m-1)(m-2)}{2}+1$ components, which shows that his estimate cannot be improved without introducing new ingredients.

It is natural to ask whether there exists a similar inequality for surfaces in the three-dimensional projective space. This question is known as the Harnack problem. Understood literally, *i.e.* as a question about the number of components, it has appeared to be a difficult problem. The maximal number of components is found only for degree ≤ 4. However Harnack Inequality has been generalized in another way.

Theorem 1.1 (Generalized Harnack Inequality)**.** *If X is a real algebraic variety, then*

$$\dim_{\mathbb{Z}_2} H_*(\mathbb{R}X;\mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_*(\mathbb{C}X;\mathbb{Z}_2), \tag{1}$$

where $\mathbb{R}X$ and $\mathbb{C}X$ are the sets of real and complex points of X, respectively.

Since $\mathbb{R}X$ is the fixed point set of involution $conj : \mathbb{C}X \to \mathbb{C}X$, Theorem 1.1, in turn, is a special case of the following theorem.

Key words and phrases. M-hypersurfaces, Hodge numbers, combinatorial patchworking, tropical hypersurfaces.

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Theorem 1.2 (Smith - Floyd Inequality)**.** *Let X be a topological space, $\tau : X \to X$ an involution and F the fixed point set of τ. Then*

$$\dim_{\mathbb{Z}_2} H_*(F; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_*(X; \mathbb{Z}_2).$$

(See, e. g., Bredon [Bre72]. To avoid a discussion of choice of homology theory, one can suppose that X and τ are simplicial.)

Although Theorem 1.2 was first stated by E. E. Floyd [Flo52], all arguments needed for the proof appeared in earlier works by P. A. Smith, see [Smi38]. Theorem 1.1 was formulated first by R. Thom [Tho65]. He got the inequality (1) as a corollary of Theorem 1.2. He did not observe however that the inequality (1) gives the best estimates of $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}X; \mathbb{Z}_2)$. It was V. M. Kharlamov [Kha72] and V. A. Rokhlin [Rok72] who acknowledged the strength and importance of Generalized Harnack Inequality. They turned the Smith theory into a powerful tool for studying the topology of real algebraic varieties ([Kha72], [Kha73], [Kha75], [Rok72]).

If X is a nonsingular curve of degree m, then $\mathbb{C}X$ is homeomorphic to a sphere with $(m-1)(m-2)/2 = (m^2 - 3m + 2)/2$ handles, and

the right hand side of the inequality (1) is $m^2 - 3m + 4$. In this case the left hand side is the doubled number of components of $\mathbb{R}X$. Hence Theorem 1.1 generalizes Harnack Inequality.

A real algebraic variety for which the left and right hand sides of the inequality (1) are equal is called an *M-variety* or a *maximal variety.*

In [IV] we proved the following statement.

Theorem 1.3. *For any positive integers m and n, there exists a nonsingular hypersurface X of degree m in $\mathbb{R}P^n$ such that*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X; \mathbb{Z}_2) = \sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X),$$

where $h^{p,q}$ are Hodge numbers.

In particular, for any positive integers m and n, there exists an M-hypersurface of degree m in $\mathbb{R}P^n$. Notice that, for a nonsingular hypersurface X of degree m in $\mathbb{C}P^n$, one has $\sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X) = h^{p,n-1-p}(\mathbb{C}X)$ if $2p = n-1$, and $\sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X) = h^{p,n-1-p}(\mathbb{C}X) + 1$ otherwise.

The construction presented in [IV] can be seen as a combinatorial version of the construction mentioned in [Vir79a]. The M-hypersurfaces in [IV] are constructed using the *primitive patchworking.* It is a particular case of the *combinatorial patchworking*, which in turn is a particular case of the Viro method of construction of real algebraic varieties, see [Vir83], [Vir84], [Vir94], [Ris92], [Stu94], [IV96], and Section 2 below. The combinatorial patchworking provides piecewise-linear models of hypersurfaces. In the case of the primitive patchworking, these models are *nonsingular real tropical hypersurfaces* (*cf.* [Mi05]). A nonsingular algebraic hypersurface X in $\mathbb{R}P^n$ constructed by means of the primitive patchworking is called a *primitive T-hypersurface.*

Let n be a positive integer, P a real polynomial of degree n in one variable, b a vector $(b_0, b_1, \dots, b_{n-1})$ in $\mathbb{Z}^n$, and $\mathcal{C}$ a class of algebraic hypersurfaces in $\mathbb{R}P^n$. We say that $\mathcal{C}$ *satisfies the condition* $b \overset{n}{\leq} P$ (respectively, $b \overset{n}{\geq} P$) if there exists a real univariate polynomial Q of degree $n-1$ such that, for any hypersurface X in $\mathcal{C}$, one has the inequality $\sum_{p=0}^{n-1} b_i \dim_{\mathbb{Z}_2} H_p(\mathbb{R}X;\, \mathbb{Z}_2) \leq P(m) + Q(m)$ (respectively, $\sum_{p=0}^{n-1} b_i \dim_{\mathbb{Z}_2} H_p(\mathbb{R}X;\, \mathbb{Z}_2) \geq P(m) + Q(m)$), where m is the degree of X. We say that $\mathcal{C}$ satisfies the condition $b \overset{n}{=} P$ if $\mathcal{C}$ satisfies the both conditions $b \overset{n}{\leq} P$ and $b \overset{n}{\geq} P$.

Let $B \in \mathbb{Z}^n$ be the vector with all the coordinates equal to 1. Since for any nonsingular hypersurface X of degree m in $\mathbb{C}P^n$ one has

$$\dim_{\mathbb{Z}_2} H_*(\mathbb{C}X; \mathbb{Z}_2) = \frac{(m-1)^{n+1} - (-1)^{n+1}}{m} + n + (-1)^{n+1}$$

(see, for example, [Far57]), the Generalized Harnack Inequality implies that the class of all nonsingular algebraic hypersurfaces in $\mathbb{R}P^n$ verifies the condition $B \overset{n}{\leq} x^n$. We say that a sequence $(X_m)_{m\in\mathbb{N}}$, where X_m is a nonsingular hypersurface of degree m in $\mathbb{R}P^n$, is *asymptotically maximal*, if this sequence verifies the condition $B \overset{n}{=} x^n$, *i.e.*, if $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}X_m;\, \mathbb{Z}_2) = m^n + O(m^{n-1})$.

For any integer $p = 0, \dots, n-1$, put

$$H_p(x) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \binom{x(p+1) - (x-1)i - 1}{n}.$$

If X is a nonsingular hypersurface of degree m in $\mathbb{C}P^n$, then $H_p(m) = h^{p,n-1-p}(\mathbb{C}X) - 1$ in the case $n-1 = 2p$, and $H_p(m) = h^{p,n-1-p}(\mathbb{C}X)$ otherwise (see [DKh86]).

For any integer $p = 0, \dots, n-1$, denote by B_p the vector in $\mathbb{Z}^n$ such that all the coordinates of B_p are equal to 0 except the p-th coordinate which is equal to 1.

The main result of the present paper is the following theorem.

Theorem 1.4. *For any positive integer n and any integer $p = 0, \dots, n-1$, the class of primitive T-hypersurfaces in $\mathbb{R}P^n$ satisfies the condition $B_p \overset{n}{\leq} H_p$.*

Theorem 1.4 immediately implies the following statement.

Corollary 1.5. *For any positive integer n, any integer $p = 0, \dots, n-1$, and any asymptotically maximal sequence $(X_m)_{m\in\mathbb{N}}$ such that X_m is a primitive T-hypersurface of degree m in $\mathbb{R}P^n$, the sequence $(X_m)_{m\in\mathbb{N}}$ satisfies the condition $B_p \overset{n}{=} H_p$, i.e.,*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m;\, \mathbb{Z}_2) = h^{p,n-1-p}(\mathbb{C}X_m) + O(m^{n-1}).$$

Remark 1.1. As it was shown by B. Bertrand [Ber06], for any primitive T-hypersurface X in $\mathbb{R}P^n$ (the projective space can be replaced by any nonsingular projective toric variety) the Euler characteristic of $\mathbb{R}X$ is equal to the signature of $\mathbb{C}X$.

Remark 1.2. The statement of Theorem 1.4 (and the statement of Corollary 1.5) becomes false if one replaces the class of primitive T-hypersurfaces by the class of all nonsingular algebraic hypersurfaces in $\mathbb{R}P^n$. For example, there exists an asymptotically maximal sequence $(Y_m)_{m\in\mathbb{N}}$ of nonsingular surfaces in $\mathbb{R}P^3$ such that Y_m is of degree m and $\dim_{\mathbb{Z}_2} H_0(\mathbb{R}Y_m;\mathbb{Z}_2) = \frac{7}{24}m^3+O(m^2)$; see [Vir79b] (note that $h^{0,2}(\mathbb{C}Y_m) = \frac{1}{6}m^3+O(m^2)$). More detailed information concerning the asymptotic behavior of Betti numbers of algebraic hypersurfaces in $\mathbb{R}P^n$ can be found in [Bih03].

The paper is organized as follows. Section 2 is devoted to the combinatorial patchworking. The key upper bounds used in the proof of Theorem 1.4 are based on the results of [Sh96, ISh03] and are presented in Sections 3 and 4. These upper bounds together with a combinatorial description of Hodge numbers of algebraic hypersurfaces in $\mathbb{C}P^2$ (Corollary 5.2) give a proof of Theorem 1.4. The combinatorial description of Hodge numbers is proved in Sections 5 - 8. Section 5 contains a construction of an asymptotically maximal sequence of hypersurfaces in $\mathbb{R}P^n$. This construction is a simplified version of the construction described in [IV] (the latter construction produces maximal hypersurfaces). To prove that the constructed sequence of hypersurfaces satisfies the condition $B_p \overset{n}{=} H_p$, we present a collection of cycles of these hypersurfaces (Section 6), and prove a recurrent relation for the Hodge numbers (Section 7).

2. Combinatorial Patchworking of Hypersurfaces in $\mathbb{R}P^n$

Let m be a positive integer number (it would be the degree of the hypersurface under construction) and $T^n(m)$ be the simplex in $\mathbb{R}^n$ with vertices $(0,0,\dots,0)$, $(0,0,\dots,0,m)$, $(0,\dots,0,m,0)$, $\dots$, $(m,0,\dots,0)$. We shorten the notation of $T^n(m)$ to T, when n and m are unambiguous and call $T^n(m)$ the *standard n-simplex of size m*. Take a triangulation τ of T with vertices having integer coordinates. Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates $(i_1,\dots,i_n)$ is denoted by $\alpha_{i_1,\dots,i_n}$.

Denote by T_* the union of all the symmetric copies of T under reflections and compositions of reflections with respect to coordinate hyperplanes. Extend the triangulation τ to a symmetric triangulation τ_* of T_*, and the distribution of signs $\alpha_{i_1,\dots,i_n}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate hyperplane we preserve its sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If an n-simplex of the triangulation of T_* has vertices of different signs, select a piece of hyperplane being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by Γ the union of the selected pieces. It is a piecewise-linear hypersurface contained in T_*. It is not a simplicial subcomplex of T_*, but can be deformed by an isotopy preserving τ_* to a subcomplex K of the first barycenter subdivision τ_*' of τ_*. Each n-simplex of τ_*' has a unique vertex belonging to τ_*. Denote by τ_*^+ the union of all the n-simplices of τ_*' containing positive vertices of τ_* and by τ_*^- the union of all the rest n-simplices. The subcomplex K is the intersection of τ_*^+ and τ_*^-. A point of Γ

contained in a simplex σ of τ_* belongs to a unique segment connecting the face of σ with positive vertices and the face with negative ones. This segment meets K also in a unique point and the deformation of Γ to K can be done along those segments.

Identify by the symmetry with respect to the origin the faces of T_*. The quotient space $\widetilde{T}$ is homeomorphic to the real projective space $\mathbb{R}P^n$. Denote by $\widetilde{\Gamma}$ the image of Γ in $\widetilde{T}$.

A triangulation τ of T is said to be *convex* if there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the n-simplices of τ. Sometimes, such triangulations are also called coherent (see [GKZ94]) or regular (see [Zie94]).

Theorem 2.1 (see [Vir83], [Vir94])**.** *If τ is convex, there exists a nonsingular hypersurface X of degree m in $\mathbb{R}P^n$ and a homeomorphism $\mathbb{R}P^n \to \widetilde{T}$ mapping the set of real points $\mathbb{R}X$ of X onto $\widetilde{\Gamma}$.*

A hypersurface defined by a polynomial

$$\sum_{(i_1,\dots,i_n)\in V} \alpha_{i_1,\dots i_n} x_0^{m-i_1-\dots i_n} x_1^{i_1} \dots x_n^{i_n} t^{\nu(i_1,\dots i_n)},$$

where V is the set of vertices of τ, and t is positive and sufficiently small, satisfies the properties described in Theorem 2.1. The polynomial above and its affine version

$$P_t^{\nu,\alpha}(x_1,\dots,x_n) = \sum_{(i_1,\dots,i_n)\in V} \alpha_{i_1,\dots i_n} x_1^{i_1} \dots x_n^{i_n} t^{\nu(i_1,\dots i_n)},$$

are called *T-polynomials* associated with the function ν and the distribution of signs $\alpha : V \to \mathbb{R}$, $\alpha(i_1,\dots,i_n) = \alpha_{i_1,\dots,i_n}$. The hypersurface X defined by a T-polynomial is called a *T-hypersurface.* If the triangulation τ is *primitive* (that is, each n-simplex of τ is of volume $\frac{1}{n!}$), then X is called a *primitive T-hypersurface.*

3. Critical Points of T-polynomials

To any orthant O in $\mathbb{R}^n$ we associate the map $\mathcal{S}_O = s_{(i_1)} \circ s_{(i_2)} \circ \dots \circ s_{(i_k)}$, where i_1, …, i_k are the indices of all negative coordinates of a point in the interior of O, and $s_{(i_j)}$, $j = 1$, …, k, is the reflection with respect to the i_j-th coordinate hyperplane in $\mathbb{R}^n$.

Let q be a point in $\mathbb{R}^n$. An n-dimensional lattice simplex δ in an orthant O of $\mathbb{R}^n$ is called *q-generic* if the point $\mathcal{S}_O(q)$ belongs neither to δ, nor to any hyperplane containing an $(n-1)$-dimensional face of δ. Let $\delta \subset O$ be a q-generic simplex. An $(n-1)$-dimensional face of δ is called *q-visible* (resp., *non-q-visible*) if the cone over this face with the vertex at $\mathcal{S}_O(q)$ does not intersect (resp., does intersect) the interior of δ. The q-index $i^q(\delta)$ of δ is the number of q-visible $(n-1)$-dimensional faces of δ. The co-q-index of δ is the number $n - i^q(\delta)$. Denote by $V_+^q(\delta)$ (resp., $V_-^q(\delta)$) the set of vertices of δ which belong to all q-visible (resp., non-q-visible) $(n-1)$-dimensional faces of δ. A q-generic simplex δ whose vertices are equipped with signs is called *real q-critical* if all the vertices in $V_+^q(\delta)$ have the same sign and the vertices in $V_-^q(\delta)$ have the sign opposite to that of the vertices in $V_+^q(\delta)$.

A triangulation τ of $T^n(m)$ is called q-generic, if all its n-simplices are q-generic. A simplex of τ is called *q-terminal* if it is contained in an $(n-1)$-dimensional non-q-visible face of $T^n(m)$. Associate to any non-q-terminal simplex σ of τ an n-simplex of τ in the following way. Let v be a point in the relative interior of σ. Take a point $\hat{v}$ such that

- $\hat{v}$ belongs to the ray which starts at q and passes through v,
- the distance between q and $\hat{v}$ is greater than the distance between q and v,
- the segment joining v and $\hat{v}$ is contained in an n-simplex of τ.

The latter n-simplex does not depend on the choice of v in the relative interior of σ and is called the *q-upper* simplex of σ. A T-polynomial of degree m is called q-generic, if the corresponding triangulation of $T^n(m)$ is q-generic.

Theorem 3.1 (see [Sh96, ISh03]). *Let $q = (-q_1, \dots, -q_n)$ be a point with negative integer coordinates in $\mathbb{R}^n$, and $P_t^{\nu,\alpha}$ a (non-homogeneous) q-generic T-polynomial of degree m in n variables. Then, there is a one-to-one correspondence between the real critical points in $(\mathbb{R}^*)^n$ of the polynomial*

$$x_1^{q_1} \dots x_n^{q_n} P_t^{\nu,\alpha}(x_1, \dots, x_n)$$

and the real q-critical n-simplices of τ_* (where τ is the triangulation defined by ν) such that the index of a real critical point of $P_t^{\nu,\alpha}$ with positive (resp., negative) critical value is equal to the q-index (resp., co-q-index) of the corresponding simplex. If τ is primitive, each n-simplex of τ has exactly one real critical symmetric copy in τ_*.

Proposition 3.2. *Let q be a point in $\mathbb{R}^n$, and τ_1, τ_2 convex primitive q-generic triangulations of a standard simplex $T^n(m)$. Then, for any integer $i = 1, \dots, n$, the numbers of simplices of q-index i in τ_1 and in τ_2 coincide.*

Proof. As is known (see, for example, [Dai00, Ber06]), for any integer $j = 0, \dots, n$, the numbers of j-dimensional simplices in τ_1 and τ_2 coincide. Thus, the numbers S_1^j and S_2^j of j-dimensional non-q-terminal simplices in τ_1 and τ_2, respectively, also coincide. For any j-dimensional non-q-terminal simplex σ in τ_k, $k = 1, 2$, the q-index of the q-upper simplex of σ is at least $n - j$. Denote by $C_{i,1}$ and $C_{i,2}$ the numbers of n-simplices of q-index i in τ_1 and τ_2, respectively. Since $C_{n,k} = S_k^0$, $k = 1, 2$, we obtain $C_{n,1} = C_{n,2}$. Furthermore, for any integer $j = 1, \dots, n-1$, we have $C_{n-j,k} = S_k^j - \sum_{s=0}^{j-1} \binom{n-s}{j-s} C_{n-s,k}$, $k = 1, 2$. Thus, $C_{i,1} = C_{i,2}$ for any integer number $i = 1, \dots, n$. □

4. Upper Bounds for Betti Numbers of Primitive T-hypersurfaces

Let $\mathcal{P}$ be the product of a real polynomial of degree m in n variables and any monomial in n variables. Let $X_{\mathcal{P}} \subset \mathbb{C}P^n$ be the projective closure of $\{\mathcal{P} = 0\} \cap (\mathbb{C}^*)^n$. Assume that $\mathcal{P}$ has only nondegenerate critical points in $(\mathbb{R}^*)^n$ and that the hypersurface $X_{\mathcal{P}}$ is nonsingular. Denote by c_p^+ (respectively, c_p^-) the number of real critical points of $\mathcal{P}$ in $(\mathbb{R}^*)^n$ of index p and with positive (respectively, negative) critical value.

The following statement is well known and can be found, for example, in [ISh03].

Proposition 4.1. *There exists a real univariate polynomial R of degree $n-1$ such that, for any polynomial $\mathcal{P}$ as above and any integer number $p = 0, 1, \ldots, n-1$, the following inequality holds:*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_{\mathcal{P}}; \mathbb{Z}_2) \le c_p^- + c_{n-p}^+ + R(m).$$

Let

- $q \in \mathbb{R}^n$ be a point with negative integer coordinates,
- τ a convex primitive q-generic triangulation of $T^n(m)$,
- $\nu : T^n(m) \to \mathbb{R}$ a convex piecewise-linear function certifying the convexity of τ,
- α a distribution of signs at the integer points of $T^n(m)$,
- $P_t^{\nu,\alpha}$ a (non-homogeneous) T-polynomial associated with ν and α,
- X a hypersurface of degree m in $\mathbb{R}P^n$ defined by (the homogenization of) $P_t^{\nu,\alpha}$.

Denote by $C_i(m)$ the number of n-simplices of τ of q-index i. Theorem 3.1 and Proposition 4.1 imply the following statement.

Theorem 4.2 (*cf.* [Sh96, ISh03])**.** *For any integer $p = 0, \ldots, n-1$, the following inequality holds:*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X; \mathbb{Z}_2) \le C_{n-p}(m) + R(m),$$

where R is a polynomial described in Proposition 4.1.

According to Proposition 3.2, the numbers $C_n(m), \ldots, C_1(m)$ do not depend on the choice of a convex primitive q-generic triangulation τ of $T^n(m)$. To prove Theorem 1.4, it remains to compare the numbers $C_n(m), \ldots, C_1(m)$ with the numbers $H_0(m), \ldots, H_{n-1}(m)$.

If $\{X_m\}_{m\in\mathbb{N}}$ is an asymptotically maximal sequence of primitive T-hypersurfaces, then due to Theorem 4.2 and the equality $\sum_{p=0}^{n-1} C_{n-p}(m) = m^n$, we obtain

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m; \mathbb{Z}_2) = C_{n-p}(m) + O(m^{n-1}),$$

for any integer $p = 0, \ldots, n-1$. In the remaining part of the paper, we construct an asymptotically maximal sequence of primitive T-hypersurfaces and show the equality

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m; \mathbb{Z}_2) = H_p(m) + O(m^{n-1})$$

for the hypersurfaces X_m of the sequence.

5. Triangulation and Signs Generating Asymptotically Maximal Sequence of Hypersurfaces

In this section we describe for each positive integer n and for each positive integer m a triangulation $\tau^n(m)$ of the standard simplex $T^n(m)$ and a distribution of signs at the vertices of $\tau^n(m)$ which provide via Theorem 2.1 an asymptotically maximal sequence of hypersurfaces in $\mathbb{R}P^n$.

To construct the triangulation $\tau^n(m)$, we use induction on n. If $n = 1$, the triangulation $\tau^1(m)$ of $[0, m]$ is formed by m intervals $[0, 1], \ldots, [m-1, m]$ for any m.

Assume that the triangulations of the standard simplices of dimensions less than n and all the sizes are constructed and consider the n-dimensional standard simplex $T^n(m)$ of size m.

Denote by $x_1, \dots, x_n$ the coordinates in $\mathbb{R}^n$. Let $T_j^{n-1} = T^n(m) \cap \{x_n = m - j\}$, and T_j be the image of T_j^{n-1} under the orthogonal projection to the coordinate hyperplane $\{x_n = 0\}$. Numerate the vertices of each simplex $T_1, \dots, T_m$ as follows: assign 1 to the vertex at the origin and $i+1$ to the vertex with nonzero coordinate at the i-th place. Assign to the vertices of $T_1^{n-1}, \dots, T_{m-1}^{n-1}$ the numbers of their projections. A triangulation of each simplex $T_1, \dots, T_m$ is already constructed. Take the corresponding triangulations in the simplices T_j^{n-1}, if $m-j$ is even. If $m-j$ is odd, take the linear map $T_j^{n-1} \to T_j$ sending the i-th vertex of T_j^{n-1} to the vertex number $n+1-i$ of T_j $(i = 1, \dots, n)$. The preimages of simplices of the triangulation of T_j form a triangulation of T_j^{n-1}.

Let l be a nonnegative integer not greater than $n-1$. If $m-j$ is even, denote by T_j^l the l-face of T_j^{n-1} which is the convex hull of the vertices with numbers $1, \dots, l+1$. If $m-j$ is odd, denote by T_j^l the l-face of T_j^{n-1} which is the convex hull of the vertices with numbers $n-l, \dots, n$.

Now for any integer $0 \le j \le m-1$ and any integer $0 \le l \le n-1$, take the join $T_{j+1}^l * T_j^{n-1-l}$. The triangulations of T_{j+1}^l and T_j^{n-1-l} constructed by the inductive assumption define a triangulation of $T_{j+1}^l * T_j^{n-1-l}$. This gives rise to the desired triangulation $\tau^n(m)$ of $T^n(m)$. It easy to see that $\tau^n(m)$ is convex: a convex piecewise-linear function certifying the convexity of $\tau^n(m)$ can be obtained combining the following functions:

- a convex piecewise-linear function whose domains of linearity are the convex hulls of T_j^{n-1} and T_{j+1}^{n-1}, $j = 0, \dots, m-1$;
- affine-linear functions $L_j(\varepsilon) : T_j^{n-1} \to \mathbb{R}$ (here j runs over all the integers $1 \le j \le m$ such that $m-j$ is even, and ε is a sufficiently small positive number); any function $L_j(\varepsilon)$ sends a vertex with number i of T_j^{n-1} to εi;
- convex piecewise-linear functions (multiplied by appropriate constants) certifying the convexity of the triangulations of $T_1^{n-1}, \dots, T_m^{n-1}$.

The distribution of signs at the vertices of $\tau^n(m)$ is as follows: all the vertices get the sign "+".

Let $(X_m)_{m \in \mathbb{N}}$ be the sequence of hypersurfaces in $\mathbb{R}P^n$ provided according to Theorem 2.1 by the triangulations $\tau^n(m)$ and the distribution of signs described above.

Theorem 5.1. *For any positive integer n and any integer $p = 0, \dots, n-1$, the sequence $(X_m)_{m \in \mathbb{N}}$ satisfies* $\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m; \mathbb{Z}_2) = H_p(m) + O(m^{n-1})$.

Corollary 5.2. *For any positive integer n and any integer $p = 0, \dots, n-1$, one has* $H_p(m) = C_{n-p}(m) + O(m^{n-1})$.

Proof. The statement immediately follows from Theorems 5.1 and 4.2 and the equality $\sum_{p=0}^{n-1} C_{n-p}(m) = m^n$. □

Theorem 5.1 is proved in Section 8. We precede the proof by a description of a certain collection of cycles of $\mathbb{R}X_m$ (Section 6) and a recurrent relation for the Hodge numbers of algebraic hypersurfaces in $\mathbb{C}P^n$ (Section 7).

6. Narrow Cycles

For any positive integers n and m, and any integer $p = 0, \dots, n-1$, we define a collection c_i, $i \in I^{n,p}(m)$ of p-cycles of $\widetilde{\Gamma}^n(m) \subset \widetilde{T}$, where $T = T^n(m)$ and $\widetilde{\Gamma}^n(m)$ is the piecewise-linear hypersurface provided by the triangulation $\tau^n(m)$ and the distribution of signs described in Section 5 (in fact, any c_i is also a p-cycle of the hypersurface $\Gamma^n(m) \subset T_*$). The cycles c_i are called *narrow*.

The collection of narrow cycles c_i is constructed together with a collection of *axes* b_i. Any axis b_i is a $(n-1-p)$-cycle in $\widetilde{T} \setminus \widetilde{\Gamma}^n(m)$ (where p is the dimension of c_i) composed by simplices of the triangulation $\tau^n_*(m)$ of T_* and representing a homological class such that its linking number with any p-dimensional narrow cycle c_k is δ_{ik}.

Let us fix some notations. For any simplex T^l_j (where $1 \le j \le m$ and $0 \le l \le n-1$), denote by $(T^l_j)_*$ the union of the symmetric copies of T^l_j under the reflections with respect to coordinate hyperplanes $\{x_i = 0\}$, where $i = 1, \dots, l$, if $m - j$ is even, and $i = n-l, \dots, n-1$, if $m - j$ is odd, and compositions of these reflections.

Any simplex T^l_j is naturally identified with the standard simplex $T^l(j)$ in $\mathbb{R}^l$ with vertices $(0, \dots, 0), (j, 0, \dots, 0), \dots, (0, \dots, 0, j)$ via the linear map $\mathcal{L}^l_j : T^l_j \to T^l(j)$ sending

(1) the vertex with number i of T^l_j to the vertex of $T^l(j)$ with the same number, if $m - j$ is even,
(2) the vertex with number i of T^l_j to the vertex of $T^l(j)$ with the number $i-n+l+1$, if $m - j$ is odd.

It is easy to see that $\mathcal{L}^l_j$ is simplicial with respect to the chosen triangulations of T^l_j and $T^l(j)$. The natural extension of $\mathcal{L}^l_j$ to $(T^l_j)_*$ identifies $(T^l_j)_*$ with $(T^l(m))_*$ and respects the chosen triangulations.

By *a symmetry* we mean a composition of reflections with respect to coordinate hyperplanes. Let $s_{(i)}$ be the reflection of $\mathbb{R}^n$ with respect to the hyperplane $\{x_i = 0\}$, $i = 1, \dots, n$. Denote by s^l_j the symmetry of $(T^{l+1}_j)_*$ which is identical if $m - j$ is even, and coincides with the restriction of $s_{(n-l-1)} \circ \dots \circ s_{(n-1)}$ on $(T^{l+1}_j)_*$ if $m - j$ is odd.

The narrow cycles and their dual cycles are defined below using induction on n. For $n = 1$ and $m \ge 3$, the narrow cycles are the pairs of points

$$(-1/2, -3/2), \dots, (-(2m-5)/2, -(2m-3)/2),$$

(The set of narrow cycles is empty if $n = 1$ and $m = 1, 2$.) The axes are pairs of vertices

$$(-1, -m+1), (-2, -m), (-3, -m+1), \dots, (-m+2, -m),$$

if m is even, and pairs of vertices

$$(-1, -m), (-2, -m+1), (-3, -m), \dots, (-m+2, -m),$$

if m is odd.

Assume that for all natural m and all natural $k < n$ the narrow cycles c_i in the hypersurface $\widetilde{\Gamma}^k(m) \subset \widetilde{T}^k(m)$ and the axes b_i in $\widetilde{T}^k(m) \setminus \widetilde{\Gamma}^k(m)$ are constructed. The narrow cycles of the hypersurface in $\widetilde{T}^n_m$ are divided into 3 families.

Horizontal Cycles. The initial data for constructing a cycle of the first family consist of an integer j satisfying inequality $1 \leq j \leq m-1$ and a narrow cycle of the hypersurface in $(T^{n-1}(j))_*$ constructed at the previous steps. In the copy $(T^{n-1}_j)_*$ of $(T^{n-1}(j))_*$, take the copy c of this cycle and the copy b of its axis.

There exists exactly one symmetric copy of T^0_{j+1} incident to b. It is T^0_{j+1} itself, if $m-j$ is odd, and either T^0_{j+1}, or $s_{(n-1)}(T^0_{j+1})$, if $m-j$ is even. If the sign of the symmetric copy $s(T^0_{j+1})$ of T^0_{j+1} incident to b is opposite to the sign of c, we include c in the collection of narrow cycles of $\widetilde{\Gamma}$. Otherwise take $s_{(n)}(c)$ as a narrow cycle of $\widetilde{\Gamma}$. The axis of c (resp., $s_{(n)}(c)$) is the suspension of b (resp., $s_{(n)}(b)$) with the vertex $s(T^0_{j+1})$ (resp., $s_{(n)}(s(T^0_{j+1}))$) and with the vertex $s(T^0_{j-1})$ (resp., $s_{(n)}(s(T^0_{j-1}))$).

Co-Horizontal Cycles. The initial data for constructing a cycle of the second family are the same as in the case of the horizontal cycles: the data consist of an integer j satisfying inequality $1 \leq j \leq m-1$ and a narrow cycle of the hypersurface in $(T^{n-1}(j))_*$.

In the copy $(T^{n-1}_j)_*$ of $(T^{n-1}(j))_*$, take the copy c of this cycle and the copy b of its axis. If the sign of the symmetric copy $s(T^0_{j+1})$ of T^0_{j+1} incident to b coincides with the sign of c, take b as axis of a narrow cycle of $\widetilde{\Gamma}$. Otherwise take $s_{(n)}(b)$. The corresponding narrow cycle is a suspension of c (resp., $s_{(n)}(c)$).

Join Cycles. The initial data consist of integers j and l satisfying inequalities $1 \leq j \leq m-1$, $1 \leq l \leq n-2$, the copy $c_1 \subset (T^l_{j+1})_*$ of a narrow cycle of the hypersurface in $(T^l(j+1))_*$, the copy $c_2 \subset (T^{n-1-l}_j)_*$ of a narrow cycle of the hypersurface in $(T^{n-1-l}(j))_*$ and the copies $b_1 \subset (T^l_{j+1})_*$ and $b_2 \subset (T^{n-1-l}_j)_*$ of the axes of these narrow cycles.

One of the joins $b_1 * b_2$ and $s^l_{j+1}(b_1) * s^{n-1-l}_j(b_2)$, belongs to $\tau^n_*(m)$; denote this join by J. If the signs of c_1 and c_2 coincide, take J as the axis of a cycle of $\widetilde{\Gamma}^n(m)$. Otherwise take $s_{(n)}(J)$. The corresponding narrow cycle is either $c_1 * c_2$, or $s^l_{j+1}(c_1) * s^{n-1-l}_j(c_2)$, or $s_{(n)}(c_1 * c_2)$, or $s_{(n)}(s^l_{j+1}(c_1) * s^{n-1-l}_j(c_2))$.

Proposition 6.1. *For any integer $p = 0, \ldots, n-1$, the $\mathbb{Z}_2$-homology classes of the narrow cycles c_i, $i \in I^{n,p}(m)$, are linearly independent in $H_p(\widetilde{\Gamma}^n(m); \mathbb{Z}_2)$.*

Proof. Both c_i and b_i with $i \in I^{n,p}(m)$ are $\mathbb{Z}_2$-cycles homologous to zero in $\widetilde{T}$, which is homeomorphic to the projective space of dimension n. The sum of dimensions of c_i and b_i is $n-1$. Thus we can consider the linking number of c_i, $i \in I^{n,p}(m)$, and b_k, $k \in I^{n,p}(m)$, taking values in $\mathbb{Z}_2$. Each c_i bounds an obvious ball in $\widetilde{T}$. This ball meets b_i in a single point transversally and is disjoint with b_k for $k \neq i$ and $i, k \in I^{n,p}(m)$. Hence the linking number of c_i and b_k is δ_{ik}. This proves that the cycles c_i, $i \in I^{n,p}(m)$, realize linearly independent $\mathbb{Z}_2$-homology classes of $\widetilde{\Gamma}^n(m)$. □

7. Recurrent Relation for Hodge Numbers

For positive integers n and m, and an integer $p = 0, \ldots, n-1$, denote by $A_m^{n,p}$ the number of ordered $(n+1)$-partitions of $m(p+1)$ such that each of the summands does not exceed $m-1$. In other words, this is the number of interior integer points in the section of the cube $[0,m]^{n+1}$ by the hyperplane $\sum_{i=1}^{n+1} x_i = m(p+1)$.

We have $A_m^{n,p} = h^{p,n-1-p}(\mathbb{C}X) - 1$, if $n-1 = 2p$, and $A_m^{n,p} = h^{p,n-1-p}(\mathbb{C}X)$ otherwise, where X is a nonsingular surface of degree m in $\mathbb{C}P^n$ (see [DKh86]). Furthermore,

$$A_m^{n,p} = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \binom{m(p+1)-(m-1)i-1}{n}.$$

If either $n < 0$ or $p < 0$, put $A_m^{n,p} = 0$. If $n = 0$ and $p \neq 0$, put $A_m^{n,p} = 0$. Finally, if $n = 0$ and $p = 0$, put $A_m^{n,p} = 1$.

Proposition 7.1. *Let n and m be positive integers, and p a nonnegative integer not greater than $n-1$. The following recurrent relation holds true:*

$$\begin{aligned} A_m^{n,p} = {} & \sum_{j=1}^{m-1} A_j^{n-1,p} + \sum_{j=1}^{m-1} A_j^{n-1,p-1} + \sum_{j=1}^{m-1} A_{j+1}^{n-2,p-1} + \sum_{j=1}^{m-1} A_j^{n-2,p-1} + \\ & \sum_{j=1}^{m-1}\sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A_{j+1}^{l,k} A_j^{n-1-l,p-1-k} + \sum_{j=1}^{m-1}\sum_{l=1}^{n-3}\sum_{k=0}^{p-1} A_{j+1}^{l,k} A_j^{n-2-l,p-1-k} + \\ & \sum_{j=1}^{m-1}\sum_{l=1}^{n-3}\sum_{k=0}^{p-2} A_{j+1}^{l,k} A_j^{n-2-l,p-2-k} + \sum_{j=1}^{m-1}\sum_{l=1}^{n-4}\sum_{k=0}^{p-2} A_{j+1}^{l,k} A_j^{n-3-l,p-2-k}. \end{aligned}$$

Proof. We prove the statement using induction on m. If $m = 1$ the statement is evident. For the inductive step, we need to show that

$$\begin{aligned} A_m^{n,p} = {} & A_{m-1}^{n,p} + A_{m-1}^{n-1,p} + A_{m-1}^{n-1,p-1} + A_m^{n-2,p-1} + A_{m-1}^{n-2,p-1} + \\ & \sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-1-l,p-1-k} + \sum_{l=1}^{n-3}\sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-2-l,p-1-k} + \\ & \sum_{l=1}^{n-3}\sum_{k=0}^{p-2} A_m^{l,k} A_{m-1}^{n-2-l,p-2-k} + \sum_{l=1}^{n-4}\sum_{k=0}^{p-2} A_m^{l,k} A_{m-1}^{n-3-l,p-2-k}. \end{aligned}$$

We call an ordered $(n+1)$-partition of $m(p+1)$ *appropriate*, if each of its summands does not exceed $m-1$. A partition $a_1 + \ldots + a_s$ of mr such that all the summands do not exceed $m-1$ is called *reducible*, if there exist integers k and l such that $l < s-1$ and

$$\sum_{i=1}^{l+1} a_i = m(k+1).$$

For any reducible partition, denote by L the largest $l < s-1$ such that $\sum_{i=1}^{l+1} a_i$ is divisible by m. A partition of mr such that all the summands do not exceed $m-1$ and which is not reducible is called *irreducible*.

Denote the summands of an appropriate partition by $a_1, \dots, a_{n+1}$. Let us prove that

(1) $A_{m-1}^{n,p}$ is the number of appropriate irreducible partitions with $a_1 < m-1$ and $a_{n+1} > 1$,
(2) $A_{m-1}^{n-1,p}$ is the number of appropriate irreducible partitions with $a_1 < m-1$ and $a_{n+1} = 1$,
(3) $A_{m-1}^{n-1,p-1}$ is the number of appropriate irreducible partitions with $a_1 = m-1$ and $a_{n+1} > 1$,
(4) $A_{m-1}^{n-2,p-1}$ is the number of appropriate irreducible partitions with $a_1 = m-1$ and $a_{n+1} = 1$,
(5) $\sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-1-l,p-1-k}$ is the number of appropriate reducible partitions with $a_{L+2} < m-1$ and $a_{n+1} > 1$,
(6) $\sum_{l=1}^{n-3}\sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-2-l,p-1-k}$ is the number of appropriate reducible partitions with $a_{L+2} < m-1$ and $a_{n+1} = 1$,
(7) $\sum_{l=1}^{n-3}\sum_{k=0}^{p-2} A_m^{l,k} A_{m-1}^{n-2-l,p-2-k}$ is the number of appropriate reducible partitions with $a_{L+2} = m-1$ and $a_{n+1} > 1$,
(8) $\sum_{l=1}^{n-4}\sum_{k=0}^{p-2} A_m^{l,k} A_{m-1}^{n-3-l,p-2-k} + A_m^{n-2,p-1}$ is the number of appropriate reducible partitions with $a_{L+2} = m-1$ and $a_{n+1} = 1$.

Let Π be an ordered s-partition $a_1 + \dots + a_s$ of $(m-1)r$, where $a_i \le m-2$ for $i = 1, \dots, s$. This partition defines in the following way an ordered s-partition $f(\Pi) : a'_1 + \dots + a'_s$ of mr with $a'_i \le m-1$ and an ordered $(s+1)$-partition $g(\Pi) : a''_1 + \dots + a''_{s+1}$ of mr with $a''_i \le m-1$. Let $i_1, \dots, i_{r-1}$ be the integers such that

$$\sum_{j=1}^{i_q} a_j \le (m-1)q, \quad \sum_{j=1}^{i_q+1} a_j > (m-1)q,$$

for any $q = 1, \dots, r-1$. Take $a'_i = a_i + 1$ if $i = i_q + 1$ (for some $q = 1, \dots, r-1$) or $i = s$, and $a'_i = a_i$ otherwise. Take $a''_i = a_i + 1$, if $i = i_q + 1$ (for some $q = 1, \dots, r-1$), and $a''_i = a_i$ otherwise. Take, in addition, $a''_{s+1} = 1$. Note that the partitions $f(\Pi)$ and $g(\Pi)$ are both irreducible, $a'_1 < m-1$, $a'_s > 1$, and $a''_1 < m-1$. For any irreducible ordered s-partition $\Phi : a'_1 + \dots + a'_s$ of mr such that $a'_1 < m-1$, $a'_s > 1$, and $a'_i \le m-1$, $i = 2$, $\dots$, s, there exists a unique partition Π such that $f(\Pi) = \Phi$. Indeed, let $i_1, \dots, i_{r-1}$ be the integers such that

$$\sum_{j=1}^{i_q} a'_j \le mq-1, \quad \sum_{j=1}^{i_q+1} a'_j > mq-1;$$

for any $q = 1, \dots, r-1$; take $a_i = a'_i - 1$, if $i = i_q + 1$ (for some $q = 1, \dots, r-1$) or $i = s$, and $a_i = a'_i$ otherwise. (Note that $a'_{i_q+1} > 1$, because Φ is irreducible.) Similarly, for any irreducible ordered $(s+1)$-partition $\Psi : a''_1 + \ldots + a''_{s+1}$ of mr such that $a''_1 < m-1$, $a''_{s+1} = 1$, and $a''_i \le m-1$, $i = 2, \ldots, s$, there exists a unique partition Π such that $g(\Pi) = \Psi$.

The constructions of $f(\Pi)$ and $g(\Pi)$ described above give immediately (1) and (2). To prove (3) (respectively, (4)), one can apply the construction of $f(\Pi)$ (respectively, $g(\Pi)$) to ordered $(n+1)$-partitions $a_1 + \ldots + a_{n+1}$ (respectively, to ordered n-partitions $a_1 + \ldots + a_n$) of $(m-1)(p+1)$ such that $a_1 = m-1$ and $a_i \le m-2$ for $i = 2, \ldots, n+1$ (resp., $i = 2, \ldots, n$).

The statements (5) - (8) follow from (1) - (4): to any appropriate reducible partition $a_1 + \ldots + a_{n+1}$, one can associate the irreducible partition $a_{L+2} + \ldots + a_{n+1}$. □

8. Proofs of Theorems 5.1 and 1.4

Proof of Theorem 5.1. For positive integers n and m, and an integer $p = 0, \ldots, n-1$, denote by $N^{n,p}_m$ the number of narrow p-cycles c_i, $i \in I^{n,p}(m)$ constructed in Section 6. If either $n \le 0$ or $p < 0$, put $N^{n,p}_m = 0$. If $n = 0$ and $p \ne 0$, put $N^{n,p}_m = 0$. Finally, if $n = 0$ and $p = 0$, put $N^{n,p}_m = 1$.

According to the construction of narrow cycles, the numbers $N^{n,p}_m$ satisfy the following recurrent relation:

$$N^{n,p}_m = N^{n,p}_{m-1} + N^{n-1,p}_{m-1} + N^{n-1,p-1}_{m-1} + \sum_{l=1}^{n-2}\sum_{k=0}^{p-1} N^{l,k}_m N^{n-1-l,p-1-k}_{m-1}.$$

In addition, $N^{1,0}_1 = N^{1,0}_2 = 0$ and $N^{1,0}_m = m-2$ for any integer $m \ge 3$.

Fix a positive integer n and an integer $p = 0, \ldots, n-1$. Notice that $A^{n,p}_m \le (m-1)^n$ for any positive integer m. Thus, Proposition 7.1 implies that, for $n \ge 2$, one has

$$A^{n,p}_m = A^{n,p}_{m-1} + A^{n-1,p}_{m-1} + A^{n-1,p-1}_{m-1} + \sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A^{l,k}_m A^{n-1-l,p-1-k}_{m-1} + O(m^{n-2}).$$

In addition, $A^{1,0}_m = m-1$ for any positive integer m. Comparing the two recurrent relations, we obtain $N^{n,p}_m = A^{n,p}_m + O(m^{n-1})$. This proves Theorem 5.1, since, according to Proposition 6.1, the cycles c_i, $i \in I^{n,p}(m)$, realize linearly independent $\mathbb{Z}_2$-homology classes in $H_p(\widetilde{\Gamma}^n(m); \mathbb{Z}_2)$. □

Proof of Theorem 1.4. The statement immediately follows from Theorem 4.2 and Corollary 5.2. □

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Deformations of scalar-flat anti-self-dual metrics and quotients of Enriques surfaces

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Abstract. In this article, we prove that a quotient of a $K3$ surface by a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action does not admit any metric of positive scalar curvature. This shows that the scalar flat anti self-dual metrics (SF-ASD) on this manifold can not be obtained from a family of metrics for which the scalar curvature changes sign, contrary to the previously known constructions of this kind of metrics on manifolds of $b^+ = 0$.

1. Introduction

One of the most interesting features of the space of anti-self-dual or self-dual (ASD/SD) metrics on a manifold is that the scalar curvature can change sign on a connected component. That means, one can possibly join two ASD metrics of scalar curvatures of opposite signs by a 1-parameter family of ASD metrics. However, this is not the case, for example for the space of Einstein metrics. There, each connected component has a fixed sign for the scalar curvature.

As a consequence, contrary to the Einstein case, most of the examples of SF-ASD metrics are constructed by first constructing a family of ASD metrics. Then showing that there are metrics of positive and negative scalar curvature in the family, and guaranteeing that there is a scalar-flat member in this family. In the $b^+ = 0$ case actually this is the only way known to construct such metrics on a 4-manifolds. This paper presents an example of a SF-ASD Riemannian 4-manifold which is impossible to obtain by this kind of techniques since it does not have a positive scalar curvature deformation.

§2 introduces the ASD manifolds and the optimal metric problem, §3 reviews the known examples of SF-ASD metrics constructed by a deformation changing the sign of the scalar curvature, §4 introduces an action on the $K3$ surface and furnish the quotient manifold with a SF-ASD metric, §5 shows that the smooth manifold defined in §4 does not admit any positive scalar curvature (PSC) or PSC-ASD metric, finally §6 includes some related examples and remarks.

Key words and phrases. Self-Dual Metrics, Spin Structures, Dirac Operator, Kähler Manifolds, Algebraic Surfaces.

2. Weyl Curvature Tensor and Optimal Metrics

Let (M, g) be an oriented Riemannian n-manifold. Then by raising the indices, the Riemann curvature tensor at any point can be viewed as an operator $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$ hence an element of $S^2\Lambda^2 M$. It satisfies the algebraic Bianchi identity hence lies in the vector space of *algebraic curvature tensors*. This space is an O_n-module and has an orthogonal decomposition into irreducible subspaces for $n \geq 4$. Accordingly the Riemann curvature operator decomposes as:

$$\mathcal{R} = U \oplus Z \oplus W$$

where

$$U = \frac{s}{2n(n-1)} g \bullet g \quad \text{and} \quad Z = \frac{1}{n-2} \overset{\circ}{Ric} \bullet g$$

s is the scalar curvature, $\overset{\circ}{Ric} = Ric - \frac{s}{n} g$ is the trace-free Ricci tensor, "$\bullet$" is the Kulkarni-Nomizu product, and W is the *Weyl Tensor* which is defined to be what is left over from the first two piece.

Let (M, g) be an oriented Riemannian manifold of dimension n. We have a linear transformation between the bundles of exterior forms called the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$. It is the unique vector bundle isomorphism between $\binom{n}{p}$-dimensional vector bundles defined by

$$\alpha \wedge (*\beta) = g(\alpha, \beta) dV_g$$

where $\alpha, \beta \in \Lambda^p, dV_g$ is the canonical n-form of g satisfying $dV_g(e_1, , e_n) = 1$ for any oriented orthonormal basis $e_1, , e_n$. $*$ is defined pointwise but it takes smooth forms to smooth forms, so induces a linear operator $* : \Gamma(\Lambda^p) \to \Gamma(\Lambda^{n-p})$ between infinite dimensional spaces. Notice that $*1 = dV_g, *dV_g = 1$ and $*^2 = (-1)^{p(n-p)} Id_{\wedge^p}$. [Bes, AHS, War]

If n is even, star operates on the middle dimension with $*^2 = (-1)^{n/2} Id_{\wedge^{n/2}}$. Moreover it is conformally invariant in dimension $n/2$: If we rescale the metric by a scalar λ, $\widetilde{g} = \lambda g$, $dV_g = \lambda^{n/2} dV_g$ so that their product remains unchanged on $n/2$-forms since the inner product on the cotangent vectors multiplied by λ^{-1}.

If $n = 2$, $*$ acts on Λ^1 or TM^* as well as TM by duality with $*^2 = -Id_{TM}$. So it defines a complex structure on a surface.

The case we are interested is $n = 4$, i.e. we have a Riemannian 4-Manifold and $* : \Lambda^2 \to \Lambda^2$ with $*^2 = Id_{\wedge^2}$ and we have eigenspaces $E_x(\pm 1)$ over each point x denoted $(\Lambda^2_\pm)_x$ and the bundle Λ^2 splits as $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. We call these bundles, bundle of *self dual , anti self dual two forms* respectively. The splitting of two forms turns out to have a great influence on the geometry of the 4-manifold because of the fact that the Riemann curvature tensor can be considered as an operator on two forms and so it also has a corresponding splitting. The decomposition of the space of two forms yields a decomposition of any operator acting on this space. In particular $W_\pm : \Lambda^2_\pm \to \Lambda^2_\pm$ called self-dual and anti-self-dual pieces of the Weyl curvature operator. And we call g to be *self-dual* (or *anti-self-dual*) *metric* if W_-(or W_+) vanishes. If one reverses the orientation,

W_- and W_+ are interchanged, so a SD manifold can be considered as an ASD manifold with the reverse orientation.

4-manifolds which support self-dual metrics are abundant. In particular [Tau] shows that for any smooth 4-manifold, the blow up $M\#k\overline{\mathbb{CP}}_2$ admits an ASD metric for sufficiently large k. Besides, there is even a connected sum theorem : under reasonable circumstances, Donaldson and Friedman proves in their seminal paper [DonFri] that the connected sums of SD manifolds are also SD. It is a pity that these deep and beautiful results of Taubes, Donaldson and Friedman are not as well known as their other results among the general geometry-topology community. Even though the result of Donaldson and Friedman is very strong, it has a small drawback. It does not tell anything about the curvature of the connected sum metric. To remedy this situation, the author proved that one can generalize their construction to the positive scalar curvature case, i.e. under the same conditions of [DonFri] the connected sums of 4-manifolds that admit PSC-SD metrics again admit PSC-SD metrics. See [Kal] for details.

Finally, some motivation on why to study SD/ASD metrics is in order. SF-ASD metrics are solutions to the *optimal metric* problem. Optimal metric problem is a struggle to find a "best" metric for a smooth manifold. Historically, the geometers are interested in constant sectional curvature spaces. As soon as these spaces are classified, there is a question of what to do with the manifolds that do not admit any constant sectional curvature. Some of them are metrized by Einstein metrics, which have constant Ricci curvature. However there are still manifolds which do not admit any Einstein metrics. At this point SF-ASD metrics come into the picture. More precisely :

Definition 2.1 ([LeOM])**.** A Riemannian metric on a smooth 4-manifold M is called an *optimal metric* if it is the absolute minimum of the L^2 norm of the Riemann Curvature tensor on the space of metrics

$$\mathcal{K}(g) = \int_M |\mathcal{R}_g|^2 dV_g.$$

Using the orthogonal decomposition it is equal to

$$\mathcal{K}(g) = \int_M \frac{s^2}{24} + \frac{|\overset{\circ}{Ric}|^2}{2} + |W|^2 \, dV_g \ .$$

On the other hand, the generalized Gauss-Bonnet Theorem and the Hirzebruch Signature Theorem express the Euler characteristic χ and signature τ respectively as

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{s^2}{24} + |W|^2 - \frac{|\overset{\circ}{Ric}|^2}{2} \, dV_g$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \; dV_g \; .$$

Combining the two in $\mathcal{K}$ gives the following expression of $\mathcal{K}$

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2 \int_M \frac{s^2}{24} + 2|W_+|^2 \; dV_g \; .$$

This yields

Proposition 2.1 ([LeOM])**.** *Let M be a smooth compact oriented 4-manifold. If M admits a SF-ASD metric then this metric is optimal. In this case all other optimal metrics are SF-ASD, too.*

For further information on the optimal metric problem, we suggest the excellent survey article [LeOM] by C. LeBrun.

3. Constructions of SF-ASD Metrics

Here we review some of the constructions for SF-ASD metrics on 4-manifolds. We begin with

Theorem 3.1 (LeBrun[LeOM])**.** *For all integers $k \geq 6$, the manifold*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k-many}$$

admits a 1-parameter family of real analytic ASD conformal metrics $[g_t]$ for $t \in [0,1]$ such that $[g_0]$ contains a metric of $s > 0$ and $[g_1]$ contains a metric of $s < 0$.

Now we are going to state *Hopf's strong maximum principle*. Before, we give a definition. Consider the differential operator $L_c = \sum_{i,j=1}^{n} a^{ij}(x_1..x_n) \frac{\partial^2}{\partial x_i \partial x_j}$ arranged so that $a^{ij} = a^{ji}$. It is called *elliptic* ([ProWei]*p56*) at a point $x = (x_1..x_n)$ if there is a positive quantity $\mu(x)$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \mu(x) \sum_{i=1}^{n} {\xi_i}^2$$

for all n-tuples of real numbers $(\xi_1..\xi_n)$. The operator is said to be uniformly elliptic in a domain Ω if the inequality holds for each point of Ω and if there is a positive constant μ_0 such that $\mu(x) \geq \mu_0$ for all x in Ω. Ellipticity of a more general second order operator is defined via its second order term. In the matrix language, the ellipticity condition asserts that the symmetric matrix $[a^{ij}]$ is positive definite at each point x.

Lemma 3.2 (Hopf's strong maximum principle , [ProWei]p64)**.** *Let u satisfy the differential inequality*

$$(L_c + h)u \geq 0 \textit{ with } h \leq 0$$

where L_c is uniformly elliptic in Ω and coefficients of L_c and h bounded. If u attains a nonnegative maximum at an interior point of Ω, then u is constant.

Corollary 3.3 (LeBrun[LeOM]). *For all integers $k \geq 6$, the connected sum $k\overline{\mathbb{CP}}_2$ admits scalar-flat anti-self-dual (SF-ASD) metrics.*[1]

Proof. Let $h_t \in [g_t]$ be a smooth family of metrics representing the smooth family of conformal classes $[g_t]$ constructed in [LeOM]. We know that the smallest eigenvalue λ_t of the Yamabe Laplacian $(\Delta + s/6)$ of the metric h_t exists, and is a continuous function of t. It measures the sign of the conformally equivalent constant scalar curvature metric [LeePar].

But the theorem(3.1) tells us that λ_0 and λ_1 has opposite signs. Then there is some $c \in [0,1]$ for which $\lambda_c = 0$. Let u be the eigenfunction corresponding to the eigenvalue 0, for the Yamabe Laplacian of h_c , i.e. $(\Delta + s/6)u = 0$. Rescale it by a constant so that it has unit integral.

Rescale the metric h_c so that it has constant scalar curvature [LeePar]. We have three cases for the scalar curvature, positive, zero or negative. If it is zero then we are done. Suppose $s_c = s > 0$. Since u is a continuous function on the compact manifold, it has a minimum, say at m. Choose the normal coordinates around there, so that $\Delta u(m) = -\sum_{k=1}^{4} \partial^2 u(m)$. Second partial derivatives are greater than or equal to zero, $\Delta u(m) \leq 0$ so $u(m) = -\frac{6}{s}\Delta u(m) \geq 0$. Assume $u(m) = 0$. Then the maximum of $-u$ is attained and it is nonnegative with $(-\Delta - s/6)(-u) = 0 \geq 0$. So the strong maximum principle (3.2) is applicable and $-u \equiv 0$, which is not an eigenfunction. So u is a positive function. For a conformally equivalent metric $\tilde{g}$, the new scalar curvature $\tilde{s}$ is computed to be [Bes]

$$\tilde{s} = 6u^{-3}(\Delta + s/6)u$$

in terms of s. Thus $\widetilde{g} = u^2 h_c$ is a scalar-flat anti-self-dual metric on $k\overline{\mathbb{CP}}_2$ for any $k \geq 6$. The negative scalar curvature case is treated similarly. □

Another construction tells us

Theorem 3.4 ([Kim]). *There exist a continuous family of self-dual metrics on a connected component of the moduli space of self-dual metrics on*

$$l(S^3 \times S^1)\# m\mathbb{CP}_2 \text{ for any } m \geq 1 \text{ and for some } l \geq 2$$

which changes the sign of the scalar curvature.

[1] Quite recently, LeBrun and Maskit announced that they have extended this result to the case $k = 5$ with similar techniques, which is the minimal number for these type of connected sums according to [LeSD].

4. SF-ASD Metric on the Quotient of Enriques Surface

In this section we are going to describe what we mean by $K3/\mathbb{Z}_2\oplus\mathbb{Z}_2$, and the scalar-flat anti-self-dual (SF-ASD) metric on it.

Let A and B be real 3×3 matrices. For $x, y \in \mathbb{C}^3$, consider the algebraic variety $V_{2,2,2} \subset \mathbb{CP}_5$ given by the equations

$$\sum_j A_i^j x_j^2 + B_i^j y_j^2 = 0 \ , \ i = 1, 2, 3$$

or more precisely,

$$\begin{aligned} A_1^1 x_1^2 + A_1^2 x_2^2 + A_1^3 x_3^2 + B_1^1 y_1^2 + B_1^2 y_2^2 + B_1^3 y_3^2 = 0 \\ A_2^1 x_1^2 + A_2^2 x_2^2 + A_2^3 x_3^2 + B_2^1 y_1^2 + B_2^2 y_2^2 + B_2^3 y_3^2 = 0 \\ A_3^1 x_1^2 + A_3^2 x_2^2 + A_3^3 x_3^2 + B_3^1 y_1^2 + B_3^2 y_2^2 + B_3^3 y_3^2 = 0 \end{aligned}$$

For generic A and B, this is a complete intersection of three nonsingular quadric hypersurfaces. By the Lefschetz hyperplane theorem, it is simply connected, and

$$K_{V_2} = K_{\mathbb{P}^5} \otimes [V_2^{\mathbb{P}^5}] = \mathcal{O}(-6) \otimes \mathcal{O}(1)^{\otimes 2} = \mathcal{O}(-4)$$

since $[V_2]_h = 2[H]_h$ and taking Poincare duals, similarly

$$K_{V_{2,2}} = K_{V_2} \otimes [V_{2,2}^{\mathbb{P}^5}] = \mathcal{O}(-4) \otimes \mathcal{O}(2) = \mathcal{O}(-2)$$

$$K_{V_{2,2,2}} = K_{V_{2,2}} \otimes [V_{2,2,2}^{\mathbb{P}^5}] = \mathcal{O}(-2) \otimes \mathcal{O}(2) = \mathcal{O}$$

finally. So the canonical bundle is trivial. V is a $K3$ Surface. We define the commuting involutions $\sigma^\pm$ by

$$\sigma^+(x, y) = (x, -y) \text{ and } \sigma^-(x, y) = (\bar{x}, \bar{y})$$

and since we arranged A and B to be real, $\sigma^\pm$ both act on V.

At a fixed point of σ^+ on V, we have $y_j = -y_j = 0$, so $\sum_j A_i^j x_j^2 = 0$. So if we take an invertible matrix A, these conditions are only satisfied for $x_j = y_j = 0$ which does not correspond to a point, so σ^+ is free and holomorphic. At a fixed point of σ^- on V, x_j's and y_j's are all real. If $A_1^j, B_1^j > 0$ for all j then $\sum_j A_i^j x_j^2 + B_i^j y_j^2 = 0$ forces $x_j = y_j = 0$ making σ^- free. At a fixed point $\sigma^-\sigma^+$ on V, $x_j = \bar{x}_j$ and $y_j = -\bar{y}_j$, so x_j's are real and y_j's are purely imaginary. Then y_j^2 is a negative real number. So if we choose $A_2^j > 0$ and $B_2^j < 0$, this forces $x_j = y_j = 0$, again we obtain a free action for $\sigma^-\sigma^+$. Thus choosing A and B within these circumstances $\sigma^\pm$ generate a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action and we define $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to be the quotient of $K3$ by this free action. We have

$$\chi = \sum_{k=0}^{4} (-1)^k b_k = 2 - 2b_1 + b_2 = 2 + (2b^+ - \tau) \text{ hence } b^+ = (\chi + \tau - 2)/2$$

so, $b^+(K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = (24/4 - 16/4 - 2)/2 = 0$, a special feature of this manifold.

Next we are going to furnish this quotient manifold with a Riemannian metric. For that purpose, there is a crucial observation [HitEin] that, for any free involution on $K3$, there exists a complex structure on $K3$ making this involution holomorphic, so the quotient is a complex manifold. We begin by stating the

Theorem 4.1 (Calabi-Yau[Cal, Yau, GHJ, Joy]). *Let (M, ω) be a compact Kähler n-manifold. Let ρ be a $(1,1)$-form belonging to the class $2\pi c_1(M)$ so that it is closed. Then, there exists a unique Kähler metric with form ω' which is in the same class as in ω, whose Ricci form is ρ.*

Intuitively, one can slide the Kähler form ω in its cohomology class and obtain any desired reasonable Ricci form ρ.

Remark 4.2. Since $c_1(K3) = 0$ in our case, taking $\rho \equiv 0$ gives us a Ricci-Flat(RF) metric on the $(K3, \omega)$ surface, the *Calabi-Yau metric.* This metric is hyperkählerian because of the following reason: The holonomy group of Kähler manifolds are a subgroup of U_2. However, Ricci-flatness reduces the holonomy since harmonic forms are parallel because of the Weitzenböck Formula for the Hodge/modern Laplacian on 2-forms (5.6). Scalar flatness and non-triviality of b^+ is to be checked. $b_1(K3) = 0$ implies $b^+(K3) = (24 - 16 - 2)/2 = 3$, which is nonzero. Actually b^+ is nontrivial for any Kähler surface since the Kähler form is harmonic and self-dual. Harmonic parallel forms are kept fixed by the holonomy group, a fact that imposes a reduction from U_2 to SU_2 which is the next possible option and isomorphic to Sp_1 in this dimension, hence the Calabi-Yau metric is hyperkähler. Alternatively one can see that the holomorphic forms are also parallel by a similar argument, another reason to reduce the holonomy.

So we have at least three almost complex structures I, J, K, parallel with respect to the Riemannian connection. By duality we regard these as three linearly independent self-dual 2-forms, parallelizing Λ_2^+. So any parallel Λ_2^+ form on $K3$ defines a complex structure after normalizing. In other words $aI + bJ + cK$ defines a complex structure for the constants satisfying $a^2 + b^2 + c^2 = 1$, i.e the normalized linear combination. On the other hand

$$b_1(K3/\mathbb{Z}_2) = b_1(K3) = 0 \ , \ b^+(K3/\mathbb{Z}_2) = (12 - 8 - 2)/2 = 1.$$

Since the pullback of harmonic forms stay harmonic, the generating harmonic 2-form on $K3/\mathbb{Z}_2$ comes from the universal cover, so is fixed by the $\mathbb{Z}_2$ action. It is also a parallel self-dual form so its normalization is then a complex structure left fixed by $\mathbb{Z}_2$. So the quotient is a complex surface with $b_1 = 0$ and $2c_1 = 0$ implying that it is an *Enriques Surface.*

So we saw that any involution or $\mathbb{Z}_2$-action can be made holomorphic by choosing the appropriate complex structure on $K3$. In particular by changing the complex structure, σ^- becomes holomorphic, too and then both $K3/\mathbb{Z}_2^{\pm}$ are complex manifolds, i.e. Enriques Surfaces, for $\mathbb{Z}_2^{\pm} = \langle \sigma^{\pm} \rangle$.

Remark 4.3. Even though we managed to make σ^+ and σ^- into holomorphic actions by modifying the complex structure, it is impossible to provide a complex structure according to which they are holomorphic at the same time. The reason is that, in such a situation the quotient $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ would be a complex manifold. On the other hand the *Noether's Formula* [Bea]

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi(S)) = \frac{1}{12}(c_1^2 + c_2)[S]$$

holds for any compact complex surface [BPV] as a consequence of the Hirzebruch-Riemann-Roch Theorem. It produces a non-integer holomorphic Euler characteristic $\frac{1}{12}\frac{24}{4} = \frac{1}{2}$.

Now consider another metric on $K3$: the restriction of the *Fubini-Study metric* on $\mathbb{CP}^5$ obtained from the Kähler form

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log|(x_1, x_2, x_3, y_1, y_2, y_3)|^2$$

We also denote the restriction metric by g_{FS}. It is clear that $\sigma^{\pm}$ leave this form invariant, hence they are isometries of g_{FS}. This is not the metric we are seeking for. This metric has all sectional curvatures lying in the interval $[1, 4]$ and is actually Einstein, i.e. $Ric = 6g$ with constant positive scalar curvature equal to 2 [Pet]p84. Let g_{RF} be the Ricci-Flat Yau metric (4.1) taking $\rho \equiv 0$ with Kähler form cohomologous to ω_{FS}. We will show that this metric is invariant under $\sigma^{\pm}$ and projects down to a metric on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Scalar flatness and being ASD are equivalent notions for Kähler metrics [LeSD], and the local structure does not change under isometric quotients which makes the quotient SF-ASD.

Since σ^+ is holomorphic, the pullback form $\sigma^{+*}\omega_{RF}$ is Kähler and the equalities

$$[\sigma^{+*}\omega_{RF}] = \sigma^{+*}[\omega_{RF}] = \sigma^{+*}[\omega_{FS}] = [\sigma^{+*}\omega_{FS}] = [\omega_{FS}]$$

show that it is cohomologous to the Fubini-Study form. Ricci curvature is preserved and is zero, hence by Calabi uniqueness (4.1) we get $\sigma^{+*}g_{RF} = g_{RF}$. Dealing with the anti-holomorphic involution needs a little more care. Think $\sigma^- : K3 \to K3$ as a diffeomorphism. The pullback of a Kähler metric is Kähler with respect to the pullback complex structure. The anti-holomorphicity relation relates the two complex structures by $\sigma^-_* J_1 = -J_2\sigma^-_*$. The pullback Kähler form $\widetilde{\omega}_n = \omega_{\sigma^{-*}g_{FS}} = -\sigma^{-*}\omega_{RF}$ since

$$\begin{aligned}\widetilde{\omega}_n(u, v) = \sigma^{-*}g_{RF}(J_1 u, v) = g_{RF}(\sigma^-_* J_1 u, \sigma^-_* v) = g_{RF}(-J_2\sigma^-_* u, \sigma^-_* v)\\ = -\omega_{RF}(\sigma^-_* u, \sigma^-_* v) = -\sigma^{-*}\omega_{RF}(u, v),\end{aligned}$$

and hence,

$$[\widetilde{\omega}_n] = [-\sigma^{-*}\omega_{RF}] = -\sigma^{-*}[\omega_{RF}] = -\sigma^{-*}[\omega_{FS}] = -[\sigma^{-*}\omega_{FS}] = -[\omega_{FS}].$$

But this is the form of $\sigma^{-*}g_{FS}$ with respect to the pullback complex structure which is the conjugate(negative) of the original one. Looking from the real point of view, once we have a Kähler metric g, it has a Kähler form corresponding to each supported complex structure on the manifold. Once the complex structure is chosen, the form is obtained by lowering an index

$$\omega_{ab} = \omega(\partial_a, \partial_b) = g(J\partial_a, \partial_b) = g(J_a^{\ c}\partial_c, \partial_b) = J_a^{\ c}\, g(\partial_c, \partial_b) = J_a^{\ c}\, g_{cb} = J_{ab}.$$

So, the form and the complex structure are equivalent from the tensorial point of view. If we conjugate(negate) the complex structure, we should replace the form with its negative.

Returning to our case, $\widetilde{\omega}_n$ is the form corresponding to the pullback, hence to the conjugate complex structure. We take its negative to obtain the one corresponding to the original complex structure. So the corresponding form is going to be $\widetilde{\omega} = - - \omega_{FS}$ which is ω_{FS} , and again the Calabi uniqueness (4.1) implies $\sigma^{-*} g_{RF} = g_{RF}$.

Remark 4.4. There is an alternative argument in [McI]p894 which appears to have a gap: "Fubini-Study metric projects down to the metrics $g^{\pm}_{FS}$ on $K3/\mathbb{Z}^{\pm}_2$. Let $h^{\pm}$ be the Calabi-Yau metric(4.1) on $K3/\mathbb{Z}^{\pm}_2$ with Kähler form cohomologous to that of $g^{\pm}_{FS}$. To remedy the ambiguity in the negative side, keep in mind that, σ^- fixes the metric and the form on $K3$, though the quotient is not a Kähler manifold initially since it is not a complex manifold, it is locally Kähler. We arrange the complex structure of $K3$ to provide a complex structure to the form, so the quotient manifold is Kähler. Now we have two Kähler metrics on the quotient (for different complex structures) but we do not know much about their curvatures, and want to make it Ricci-Flat, so we use the Calabi-Yau argument. Since $c_1(K3/\mathbb{Z}^{\pm}_2) = 0$ with real coefficients, we pass to the Calabi-Yau metric for $\rho \equiv 0$. $\pi^{\pm}$ denoting the quotient maps, the pullback metrics $\pi^{\pm *} h^{\pm}$ are both Ricci-Flat-Kähler(RFK) metrics on $K3$ with Kähler forms cohomologous to that of g_{FS}. Their Ricci forms are both zero. By the uniqueness(4.1) of the Yau metric we have $\pi^{+*} h^+ = \pi^{-*} h^-$. Hence this is a Ricci-Flat Kähler metric on $K3$ on which both $\sigma^{\pm}$ act isometrically. This metric therefore projects down to a Ricci-Flat metric on our manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$." The problem is that the pullback metrics $\pi^{\pm *} h^{\pm}$ are Kähler metrics with cohomologous Kähler forms, however they are Kählerian with respect to different complex structures. So the Calabi uniqueness (4.1) can not be applied directly.[2]

5. Weitzenböck Formulas

Now we are going to show that the smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any positive scalar curvature metric. For that purpose we state the Weitzenböck Formula for the Dirac Operator on spin manifolds. Before that we introduce some notation together with some ingredients of the formula.

For any vector bundle E over a Riemannian Manifold M, the Levi-Civita connection is going to be the linear map we denote by $\nabla : \Gamma(E) \to \Gamma(Hom(TM, E))$. Then we get the adjoint $\nabla^* : \Gamma(Hom(TM, E)) \to \Gamma(E)$ defined implicitly by

$$\int_M \langle \nabla^* S, s \rangle dV = \int_M \langle S, \nabla s \rangle dV$$

and we define the *connection Laplacian* of a section $s \in \Gamma(E)$ by their composition $\nabla^* \nabla s$. Notice that the harmonic sections are parallel for this operator. Using the metric, we can express its action as :

Proposition 5.1 ([Pet]p179)**.** *Let (M, g) be an oriented Riemannian manifold, $E \to M$ a vector bundle with an inner product and compatible connection. Then*

$$\nabla^* \nabla s = -tr \nabla^2 s$$

[2]Thanks to the second referee for pointing out this delicate issue.

for all compactly supported sections of E.

Proof. First we need to mention the second covariant derivatives and then the integral of the divergence. We set

$$\nabla^2 K(X,Y) = (\nabla\nabla K)(X,Y) = (\nabla_X \nabla K)(Y).$$

Then using the fact that ∇_X is a derivation commuting with every contraction: [KobNom]p124

$$\begin{aligned}\nabla_X\nabla_Y K &= \nabla_X C(Y\otimes\nabla K) = C\nabla_X(Y\otimes\nabla K)\\ &= C(\nabla_X Y\otimes\nabla K + Y\otimes\nabla_X\nabla K)\\ &= \nabla_{\nabla_X Y}K + (\nabla_X\nabla K)(Y)\\ &= \nabla_{\nabla_X Y}K + \nabla^2 K(X,Y)\end{aligned}$$

hence $\nabla^2 K(X,Y) = \nabla_X\nabla_Y K - \nabla_{\nabla_X Y}K$ for any tensor K. That is how the second covariant derivative is defined . Higher covariant derivatives are defined inductively.

For the divergence, remember that

$$(divX)dV_g = \mathcal{L}_X dV_g,$$

which is taken as a definition sometimes[KobNom]p281. After combining this with the Cartan's Formula: $\mathcal{L}_X dV = di_X dV + i_X d(dV) = di_X dV$; Stokes' Theorem yields that $\int_M (divX)dV = \int_M \mathcal{L}_X dV = \int_M d(i_X dV) = \int_{\partial M} i_X dV = 0$ for a compact manifold without boundary. This is actually valid even for a noncompact manifold together with a compactly supported vector field.

Now take an open set on M with an orthonormal basis $\{E_i\}_{i=1}^n$. Let s_1 and s_2 be two sections of E compactly supported on the open set. We reduce the left-hand side via multiplying by s_2 as follows:

$$\begin{aligned}(\nabla^*\nabla s_1, s_2)_{L^2} = \int_M \langle\nabla^*\nabla s_1, s_2\rangle dV = \int_M \langle\nabla s_1,\nabla s_2\rangle dV &= \int_M tr((\nabla s_1)^*\nabla s_2)dV\\ &= \sum_{i=1}^n \int_M \langle(\nabla s_1)^*\nabla s_2(E_i), E_i\rangle dV\\ &= \sum\int_M \langle(\nabla s_1)^*\nabla_{E_i}s_2, E_i\rangle dV\\ &= \sum\int_M \langle\nabla_{E_i}s_2, \nabla s_1(E_i)\rangle dV\\ &= \sum\int_M \langle\nabla_{E_i}s_1, \nabla_{E_i}s_2\rangle dV.\end{aligned}$$

Define a vector field X by $g(X,Y) = \langle\nabla_Y s_1, s_2\rangle$. Divergence of this vector field is

$$\begin{aligned}divX = -d^*(X^\flat) = tr\nabla X = \textstyle\sum_{i=1}^n\langle\nabla_{E_i}X, E_i\rangle &= \textstyle\sum(E_i\langle X, E_i\rangle - \langle X, \nabla_{E_i}E_i\rangle)\\ &= \textstyle\sum(E_i\langle\nabla_{E_i}s_1, s_2\rangle - \langle\nabla_{\nabla_{E_i}E_i}s_1, s_2\rangle).\end{aligned}$$

We know that its integral is zero, so our expression continues to evolve as

$$\sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2 \rangle dV - \int_M (divX) dV$$
$$= \sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2 \rangle dV - \sum \int_M (E_i \langle \nabla_{E_i} s_1, s_2 \rangle - \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2 \rangle) dV$$
$$= \sum \int_M (-\langle \nabla_{E_i} \nabla_{E_i} s_1, s_2 \rangle + \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2 \rangle) dV$$
$$= \sum \int_M \langle -\nabla^2 s_1(E_i, E_i), s_2 \rangle dV$$
$$= -\int_M \langle \sum \langle \nabla^2 s_1(E_i), E_i \rangle, s_2 \rangle dV$$
$$= \int_M \langle -tr \nabla^2 s_1, s_2 \rangle dV$$
$$= (-tr \nabla^2 s_1, s_2)_{L^2}.$$

So we established that $\nabla^* \nabla s_1 = -tr \nabla^2 s_1$ for compactly supported sections in an open set.

□

Theorem 5.2 (Atiyah-Singer Index Theorem[LawMic]p256,[Mor]p47)**.** *Let M be a compact spin manifold of dimension $n = 2m$. Then the index of the Dirac operator is given by*

$$ind(\not{D}^+) = \hat{A}(M) = \hat{\boldsymbol{A}}(M)[M].$$

More generally, if E is any complex vector bundle over M, the index of $\not{D}_E^+ : \Gamma(\mathbb{S}_\pm \otimes E) \to \Gamma(\mathbb{S}_\mp \otimes E)$ is given by

$$ind(\not{D}_E^+) = \{ch(E) \cdot \hat{\boldsymbol{A}}(M)\}[M].$$

For $n = 4$, $\hat{\boldsymbol{A}}(M) = 1 - p_1/24$ and the first formula reduces to

$$ind(\not{D}^+) = \hat{A}(M) = \int_M -\frac{p_1(M)}{24} = -\frac{\tau(M)}{8}$$

by the Hirzebruch Signature Theorem.

Let us explain the ingredients beginning with the cohomology class $\hat{\mathbf{A}}(M)$. Consider the power series of the following function [Fri]p108 :

$$\frac{t/2}{\sinh t/2} = \frac{t}{e^{t/2} - e^{-t/2}} = 1 + A_2 t^2 + A_4 t^4 + \dots$$

where we compute the coefficients as

$$A_2 = -\frac{1}{24}\ ,\ A_4 = \frac{7}{10 \cdot 24 \cdot 24} = \frac{7}{5760}.$$

Consider the Pontrjagin classes $p_1...p_k$ of M^{4k}. Represent these as the elementary symmetric functions in the squares of the formal variables $x_1 \cdots x_k$:

$$x_1^2 + \cdots + x_k^2 = p_1 \ , \ \cdots \ , \ x_1^2 x_2^2 \cdots x_k^2 = p_k$$

Then $\prod_{i=1}^{k} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}$ is a symmetric power series in the variables $x_1^2 \cdots x_k^2$, hence defines a polynomial in the Pontrjagin classes. We call this polynomial $\hat{\mathbf{A}}(M)$

$$\hat{\mathbf{A}}(M) = \prod_{i=1}^{k} \frac{x_i/2}{\sinh x_i/2}.$$

In lower dimensions we have

$$\hat{\mathbf{A}}(M^4) = 1 - \frac{1}{24}p_1 \ , \ \hat{\mathbf{A}}(M^8) = 1 - \frac{1}{24}p_1 + \frac{7}{5760}p_1^2 - \frac{1}{1740}p_2.$$

If the manifold has dimension $n = 4k+2$, again it has k Pontrjagin classes, and we define the polynomial $\hat{\mathbf{A}}(M^{4k+2})$ by the same formulas.

Secondly, we know that $\not{D}^+ : \Gamma(\mathbb{S}_+) \to \Gamma(\mathbb{S}_-)$ is an elliptic operator, so its kernel is finite dimensional and its image is a closed subspace of finite codimension. The *index* of an elliptic operator is defined to be $dim(kernel) - dim(cokernel)$. Actually in our case $\not{D}^+$ and $\not{D}^-$ are formal adjoints of each other: $(\not{D}^+\psi, \eta)_{L^2} = (\psi, \not{D}^-\eta)_{L^2}$ for ψ, η compactly supported sections [LawMic]p114 , [Mor]p42. Consequently the index becomes $dim(ker\not{D}^+) - dim(ker\not{D}^-)$.

This index is computed from the symbol in the following way. Consider the pullback of $\mathbb{S}_\pm$ to the cotangent bundle T^*M. The symbol induces a bundle isomorphism between these bundles over the complement of the zero section of T^*M. In this way the symbol provides an element in the relative K-theory of $(T^*M, T^*M - M)$. The Atiyah-Singer Index Theorem computes the index from this element in the relative K-theory. In the case of the Dirac operator the index is $\hat{A}(M)$, the so-called *A-hat genus* of M.

Now we are ready to state our main tool

Theorem 5.3 (Weitzenböck Formula[Pet]p183,[Bes]p55)**.** *On a spin Riemannian manifold, consider the Dirac operator* $\not{D} : \Gamma(\mathbb{S}_\pm) \to \Gamma(`S_\mp)$. *The Dirac Laplacian can be expressed in terms of the connection/rough Laplacian as*

$$\not{D}^2 = \nabla^*\nabla + \frac{s}{4}$$

where ∇ *is the Riemannian connection.*

Finally we state and prove our main result:

Theorem 5.4. *The smooth manifold* $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *does not admit any metric of positive scalar curvature(PSC).*

Proof. If $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ admits a metric of PSC then $K3$ is also going to admit such a metric because one pulls back the metric of the quotient, and obtain a locally identical metric on which the PSC survives.

So we are going to show that the $K3$ surface does not admit any metric of PSC. First of all the canonical bundle of $K3$ is trivial so that $c_1(K3) = 0 = w_2(K3)$ implying that it is a spin manifold. By the Atiyah-Singer Index Theorem (5.2),

$$ind(\not{D}^+) = \hat{\mathbf{A}}(M)[M] = -\frac{\tau(M)}{8} = 2$$

for the $K3$ Surface. Since it is equal to $dim(ker) - dim(coker)$, this implies that the $dim(ker\not{D}^+) \geq 2$.

Let $\psi \in ker\not{D}^+ \subset \Gamma(\mathbb{S}_+)$ and consider its image $(\psi, 0)$ in $\Gamma(\mathbb{S}_+ \oplus \mathbb{S}_-)$. Then $\not{D}^2(\psi, 0) = 0$ since $\not{D} = \not{D}^+ \oplus \not{D}^-$. Abusing the notation as $\psi = (\psi, 0)$ the spin Weitzenböck Formula (5.3) implies

$$0 = \nabla^*\nabla\psi + \frac{s}{4}\psi.$$

Taking the inner product with ψ and integrating over the manifold yields

$$0 = (\nabla^*\nabla\psi, \psi)_{L^2} + (\frac{s}{4}\psi, \psi)_{L^2} = (\nabla\psi, \nabla\psi)_{L^2} + \frac{s}{4}(\psi, \psi)_{L^2} = \int_M (|\nabla\psi|^2 + \frac{s}{4}|\psi|^2)dV$$

and $s > 0$ implies that $|\nabla\psi| = |\psi| = 0$ everywhere, hence $\psi \equiv 0$. So $ker\not{D}^+ = 0$, which is not the case.

Notice that $s \geq 0$ and $s(p) > 0$ for some point is also enough for the conclusion because then ψ would be parallel and zero at some point implies ψ is zero everywhere ☐

Remark 5.5. In the above proof, while taking $\psi \in ker\not{D}^+$ some confusion may arise if $ker\not{D}^+ \subset \Gamma(\mathbb{S}_+)$ is not specified. A reader might think that $\not{D}^+$ acts on $\Gamma(\mathbb{S}_+ \oplus \mathbb{S}_-)$ and ψ is equal to something like (ψ, η) for some nonzero η, so that $\not{D}^2\psi = \not{D}^+\not{D}^-\psi$.

Alternatively, we could use the Weitzenböck Formula for the Hodge/modern Laplacian to show that there are no PSC anti-self-dual(ASD) metrics on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This is a weaker conclusion though sufficient for our purposes

Theorem 5.6 (Weitzenböck Formula 2[LeOM])**.** *On a Riemannian manifold, we can express the Hodge/modern Laplacian in terms of the connection/rough Laplacian as*

$$(d + d^*)^2 = \nabla^*\nabla - 2W + \frac{s}{3}$$

where ∇ is the Riemannian connection and W is the Weyl curvature tensor.

Theorem 5.7. *The smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any anti-self-dual(ASD) metric of positive scalar curvature(PSC).*

Proof. Again we are going to show this only for $K3$ as in (5.4). Suppose we have a metric of positive scalar curvature.

Anti-self-duality reduces our Weitzenbock Formula (5.6) to the form

$$(d+d^*)^2 = \nabla^*\nabla - 2W_- + \frac{s}{3}$$

because $W = W_-$ or $W_+ = 0$.

We have already explained in (4.2) that b_2^+ of the $K3$ surface is nonzero. So take a nontrivial harmonic self-dual 2-form φ. $W_- : \Gamma(\Lambda^-) \to \Gamma(\Lambda^-)$ only acts on anti-self-dual forms, so it takes φ to zero. Applying the formula,

$$0 = \nabla^*\nabla\varphi + \frac{s}{3}\varphi$$

and taking the inner product with φ and integrating over the manifold yields similarly

$$0 = (\nabla^*\nabla\varphi, \varphi)_{L^2} + (\frac{s}{3}\varphi, \varphi)_{L^2} = (\nabla\varphi, \nabla\varphi)_{L^2} + \frac{s}{3}(\varphi, \varphi)_{L^2} = \int_M (|\nabla\varphi|^2 + \frac{s}{3}|\varphi|^2)dV$$

and $s > 0$ implies that $|\nabla\varphi| = |\varphi| = 0$ everywhere, hence $\varphi \equiv 0$, a contradiction. □

6. Other Examples

In this section, we will go through some examples. We begin with the case $b^+ = 1$.

Theorem 6.1 ([KLP],[RolSin])**.** *For all integers $k \geq 10$, the connected sum $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ admits scalar-flat-Kähler(SFK) metrics.*[3]

The case $k \geq 14$ is achieved in [KLP]. They start with blow ups of $\mathbb{CP}_1 \times \Sigma_2$ the cartesian product of rational curve and genus-2 curve, which already have a SFK metric via the hyperbolic ansatz of [LeExp]. After applying an isometric involution, they get a SFK orbifold, which has isolated singularities modelled on $\mathbb{C}^2/\mathbb{Z}_2$. Replacing these singular models with smooth ones, they obtain the desired metric.

For the case $k \geq 10$, Rollin and Singer first construct a related SFK orbifold with isolated and cyclic singularities of which the algebra $\mathfrak{a}_0$ of non-parallel holomorphic vector fields is zero. This is done by an argument analogous to that of [BurBar]. The target manifold is the minimal resolution of this orbifold. To obtain the target metric, they glue some suitable local models of SFK metrics to the orbifold. These models are asymptotically locally Euclidean(ALE) scalar flat Kähler metrics constructed in [CalSin].

When a metric is Kähler, from the decomposition of the Riemann Curvature operator, scalar-flatness turns out to be equivalent to being anti-self-dual. So these metrics are SF-ASD.

Since these manifolds have $b^+ \neq 0$ Weitzenböck Formulas apply as in section §5, so automatically the scalar curvature can not change sign. These examples show why the case $b^+ = 0$ we focussed on, is interesting.

A second type of example is [4]

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[3] It is a curious fact that $k = 10$ is the minimal number for these type of metrics(SF-ASD) on $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, known by [LeSD] long before these constructions made. See [LeOM] for a survey.

[4] Thanks to the first referee for pointing out this example and the remark.

Example 6.2. Let Σ_g be the genus-$g(>1)$ surface with Kähler metric of constant curvature $\kappa = -1$, and S^2 be the 2-sphere with the round $\kappa = +1$ metric. Consider the product metric on $S^2 \times \Sigma_g$ which is Kähler with zero scalar curvature. So this metric is anti-self-dual. Then we have fixed point free, orientation reversing, isometric involutions of both surfaces obtained by antipodal maps. Combination of these involutions yield an isometric involution on the product and the metric pushes down to a metric on

$$(S^2 \times \Sigma_g)/\mathbb{Z}_2 = \mathbb{R}P^2 \times (\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{(g+1)-many})$$

which is SF-ASD as these properties are preserved under isometry. This is an example with all the key properties ($b^+ = 0$)where the metric is completely explicit. Note that this manifold is non-orientable.

One has to be careful about the involution on Σ_g though. There are many hyperbolic metrics on Σ_g which do not have an isometric involution satisfying our conditions. Involution must be conformal. One way to achieve this is as follows. We take a conformal structure on the $(g+1)\mathbb{R}P^2$, and pull this back to its orientable double cover Σ_g. By the uniformization theorem of Riemann Surfaces, there is a unique hyperbolic($\kappa = -1$) metric of Σ_g in this conformal class. Since this metric is unique in its conformal class, it is automatically invariant under the involution and pushes down to a hyperbolic metric on $(g+1)\mathbb{R}P^2$. It is known that the moduli space of hyperbolic metrics on Σ_g is $6g-6$ real dimensional, on the other hand $3g-3$ real dimensional on $(g+1)\mathbb{R}P^2$. So it is appearent that there are many hyperbolic metrics on Σ_g which are not coming from the quotient. So that they do not have the isometric involution of the kind we use.

Another way to construct this involution can be to begin with a surface in $\mathbb{R}^3$ which is symmetric about the origin, e.g. add symmetric handles to a sphere or a torus about the origin. Then take the conformal structure induced from Euclidean $\mathbb{R}^3$. There is a unique hyperbolic metric that induces this conformal structure, so proceed as before.

Remark 6.3. The other side of the story discussed here is that we have ASD, conformally flat deformations to negative scalar curvature metrics e.g. on $M = \Sigma_g \times S^2$. It is obtained by deforming

$$\rho : \pi_1(M) \longrightarrow SL(2, \mathbb{H}),$$

the representation of the fundamental group in $SL(2, \mathbb{H})$. This is the group of conformal transformations of S^4. It contains the isometry group $SL(2, \mathbb{R})$ due to the fact that $\mathcal{H}^2 \times S^2$ is conformally flat and conformally diffeomorphic to $S^4 - S^1$. This is the universal cover of $\Sigma_g \times S^2$, and its fundamental group acts by conformal transformations in $SL(2, \mathbb{R}) \subset SL(2, \mathbb{H})$. $\pi_1(M) = \pi_1(\Sigma_g)$ is generated by $2g$ elements $\{a_1, b_1 \ \cdots \ a_g, b_g\}$ and these are subject to the single relation $\prod [a_j, b_j] = 1$ the product of the commutators. So a representation in $SL(2, \mathbb{H})$ corresponds to a choice of $2g$ elements and a relation. Since this Lie group is 15 dimensional and we have to subtract the change of basis conjugation and the relations , this kind of representations depend on $15 \times 2g - 15 \times 1 - 15 \times 1 = 30g - 30$ parameters. On the other hand, a twisted product metric on $\Sigma_g \times S^2$ provides a representation in $SL(2,\mathbb{R}) \times SO(3)$, which is a $3+3=6$ dimensional Lie group. So we have $6 \times 2g - 6 \times 1 - 6 \times 1 = 12g - 12$ parameters for this representation. This difference means that the generic conformally flat structure on M does not come from a twisted product metric. We refer [Pon] for further details.

Using the Weitzenböck Formula (5.6), LeBrun [LeSD] shows that a conformally flat metric on M of zero scalar curvature must be Kähler with respect to both orientations, and by a holonomy argument he further shows that the metric is of twisted product type.[5]

Similar parameter counts are valid for $M/\mathbb{Z}_2$ and this shows that the generic conformally flat metric on this manifold has negative Yamabe constant.

Also, by further investigation, it might be possible to get examples which are doubly covered by e.g. the simply connected examples of [KLP].

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The Kähler-Ricci flow on Kähler surfaces

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Abstract. The existence of Kähler-Einstein metrics on a compact Kähler manifold of definite or vanishing first Chern class has been the subject of intense study over the last few decades, following Yau's solution to Calabi's conjecture. The Kähler-Ricci flow is a canonical deformation for Kähler metrics. In this expository note, we apply some known results of the Kähler-Ricci flow and give a metric classification for Kähler surfaces with semi-negative or positive first Chern class.

1. Introduction

The problem of finding Kähler-Einstein metrics on a compact Kähler manifold has been the subject of intense study over the last few decades. In his solution to Calabi's conjecture, Yau [Ya1] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with vanishing or negative first Chern class. A proof of Yau's theorem is given by Cao [Ca] using the Kähler-Ricci flow.

As is well-known, Hamilton's Ricci flow has become one of the most powerful tools in geometric analysis [Ha1]. The Ricci flow can be applied to give an independent proof of the classical uniformization for Rieman surfaces (c.f. [Ha2, Ch, ChLuTi]). Recently Perelman [Pe] has made a major breakthrough in studying the Ricci flow with remarkable applications to the study of 3-manifolds. The convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds with positive first Chern class was claimed by Perelman and it has been generalized to any Kähler manifolds admitting a Kähler-Ricci soliton by Tian and Zhu with certain assumptions on the initial metrics [TiZhu]. Previously, it was proved in [ChTi] that the Kähler-Ricci flow converges to a Kähler-Einstein metric if the bisectional curvature of the initial Kähler metric is non-negative and positive at least at one point.

Most algebraic varieties do not admit Kähler-Einstein metrics, for example, those with indefinite first Chern class, so it is a natural question to ask if there exist any well-defined canonical metrics on these varieties or on their canonical models. Tsuji [Ts] applied the Kähler-Ricci flow to produce a canonical singular Kähler-Einstein metric on non-singular minimal algebraic varieties of general type. In [SoTi], new canonical metrics on the canonical models of projective varieties of positive Kodaira dimension were defined and such metrics were constructed by the Kähler-Ricci flow on Kähler surfaces.

Key words and phrases. Kähler-Ricci flow, Kähler-Einstein metrics, Chern class, Kodaira dimension.
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In this expository note, we present a metric classification for Kähler surfaces with non-negative Kodaira dimension or positive first Chern class by the Kähler-Ricci flow.

2. Preliminaries

Let X be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X. In local coordinates $(z_1, ..., z_n)$, ω can be written in the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. The Kähler form ω is a closed $(1,1)$-form by the Kähler condition. In other words, for i, j, $k = 1, ..., n$,

$$\frac{\partial g_{i\bar{k}}}{\partial z_j} = \frac{\partial g_{j\bar{k}}}{\partial z_i} \text{ and } \frac{\partial g_{k\bar{i}}}{\partial \bar{z}_j} = \frac{\partial g_{k\bar{j}}}{\partial \bar{z}_i}.$$

We also define the space of plurisubharmonic functions with respect to a Kähler form ω by

$$\mathcal{P}(X, \omega) = \{\varphi \in C^\infty(X) \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

The curvature tensor for g is locally given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l},\ i, j, k = 1, 2, ..., n.$$

The bisectional curvature of ω is positive if

$$R_{i\bar{j}k\bar{l}} v^i \bar{v}^j w^k \bar{w}^l \geq 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of X. The Ricci tensor is given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j},\ i, j = 1, 2, ..., n.$$

So its Ricci curvature form is

$$Ric(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}} dz_i \wedge d\bar{z}_j = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{k\bar{l}}).$$

Definition 2.1. *The Kähler metric ω is called a Kähler-Einstein metric on X if*

$$Ric(\omega) = \lambda\omega,$$

for some constant $\lambda \in \mathbf{R}$.

We can always scale the Kähler-Einstein metric ω so that $\lambda = -1$, 0 or 1. By the Einstein equation, the first Chern class of X has to be definite or vanishing if there exists a Kähler-Einstein metric on X.

The Ricci flow introduced by Hamilton ([Ha1]) on a Riemannian manifold is defined by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}. \tag{2.1}$$

On a Kähler manifold X, the Kähler condition is preserved by the Ricci flow if the initial metric is Kähler, so that the Ricci flow is called the Kähler-Ricci flow. We define the following normalized Kähler-Ricci flow

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} + \lambda g_{i\bar{j}}, \tag{2.2}$$

where $\lambda = -1,\ 0,\ 1$.

Let X be a compact complex manifold of complex dimension n and K_X be the canonical line bundle on X. The canonical ring of X is defined by

$$R(X) = \oplus_{m=0}^{\infty} H^0(X, K_X^m)$$

with the pairing

$$H^0(X, K_X^{m_1}) \otimes H^0(X, K_X^{m_2}) \to H^0(X, K_X^{m_1+m_2}).$$

Then we can define the Kodaira dimension $\mathrm{kod}(X)$ of X by

$$\mathrm{kod}(X) = \begin{cases} -\infty & \text{if } R(X) \cong \mathbf{C} \\ tr(R(X)) - 1 & \text{otherwise,} \end{cases}$$

where $tr(R(X))$ is the transcendental degree of the canonical ring $R(X)$.

We always have $\mathrm{kod}(X) \leq \dim(X)$. In fact, if $\mathrm{kod}(X) \geq 0$, we can define the following meromorphic pluricanonical map for sufficiently large m

$$\Phi_m : X \to \mathbf{CP}^{N_m}$$

by

$$\Phi_m(z) = [S_0^{(m)}(z), S_1^{(m)}(z), ..., S_{N_m}^{(m)}(z)],$$

where $\{S_0^{(m)}, S_1^{(m)}, ..., S_{N_m}^{(m)}\}$ is a basis of $H^0(X, K_X^m)$. The Kodaira dimension of X is exactly the complex dimension of the image of X by Φ_m for sufficiently large m.

The compact complex manifolds can be classified according to their Kodaira dimension by $\mathrm{kod}(X) = -\infty, 0, 1, ..., \dim(X)$.

In the case of smooth compact complex surfaces, the Kodaira dimension must be $-\infty$, 0, 1 or 2. We have the Enriques-Kodaira classification by Kodaira dimension dividing the minimal surfaces into ten classes. Nonsingular rational curves with self-intersection -1 are called exceptional curves of first kind or simply (-1)-curves. A smooth compact complex surface is called a minimal surface, if it does not contain any (-1)-curve. Any smooth compact complex surface is birationally equivalent to a minimal surface by contracting the (-1)-curves.

3. $\mathrm{kod}(X) = 2$

A compact complex surface X of $\mathrm{kod}(X) = 2$ is called a surface of general type. A surface of general type X is minimal if and only if its canonical line bundle K_X is semi-positive, so that $c_1(X) \leq 0$.

If $c_1(X) < 0$, K_X is a positive line bundle so that X must be a minimal algebraic surface. It is proved by Yau [Ya1] and Aubin [Au] independently that there always exists a unique Kähler-Einstein metric on X. Cao [Ca] gave an alternative proof by applying the following Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega) - \omega \\ \omega\,|_{t=0} = \omega_0 \end{cases} \tag{3.1}$$

where ω_0 is a Kähler metric on X.

Theorem 3.1 (Cao). *Let X be a minimal complex surface with $c_1(X) < 0$. Then the Kähler-Ricci flow converges for any initial Kähler metric to the unique Kähler-Einstein metric ω_{KE} with*

$$Ric(\omega_{KE}) = -\omega_{KE}.$$

In general, a minimal surface of general type might contain a finite number of (-2)-curves and $c_1(X)$ is not negative but semi-negative. Let C be the union of these curves whose connected components are rational curves of A-D-E-type (see [BHPV]). The canonical model X_{can} of X is then obtained by blowing down C

$$f : X \to X_{can}.$$

X_{can} is an orbifold surface with singularities of A-D-E-type. We can still apply the normalized Kähler-Ricci flow (3.1) on a minimal surface of general type. Let $Ka(X)$ denote the Kähler cone of X, that is,

$$K_a(X) = \{[\omega] \in H^{1,1}(X, \mathbf{R}) \mid [\omega] > 0\}.$$

The Kähler class will change along the Kähler-Ricci flow by the following ordinary differential equation

$$\begin{cases} \dfrac{\partial[\omega]}{\partial t} = -c_1(X) - [\omega] \\ [\omega]|_{t=0} = [\omega_0]. \end{cases} \tag{3.2}$$

It follows that

$$[\omega(t, \cdot)] = -c_1(X) + e^{-t}([\omega_0] + c_1(X)).$$

Let $\chi \in -c_1(X)$ be a closed semi-positive $(1,1)-$form and Ω be the smooth volume form on X such that

$$\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi.$$

We choose the reference Kähler metric ω_t by

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi)$$

so that

$$\omega = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi.$$

Then the Kähler-Ricci flow can be reduced to a Monge-Ampère flow for the potential φ given by

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \log \frac{(\omega_t+\sqrt{-1}\partial\overline{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \tag{3.3}$$

Tsuji [Ts] proved the following convergence result for the Kähler-Ricci flow.

Theorem 3.2 (Tsuji)**.** *Let X be a minimal complex surface of general type. Then the Kähler-Ricci flow converges to a unique singular Kähler-Einstein metric ω_∞ smooth outside the (-2)-curves with*

$$Ric(\omega_\infty) = -\omega_\infty$$

for any initial Kähler metric ω_0 satisfying $[\omega_0] > -c_1(X)$.

The key observation in Tsuji's proof is that the canonical bundle K_X on a minimal surface of general type is big and nef, so that for any sufficiently small $\epsilon > 0$

$$[K_X] - \epsilon[C] > 0.$$

The initial class condition is removed by Tian and Zhang [TiZha] and a stronger uniform C^0-estimate is obtained.

Theorem 3.3 (Tian-Zhang)**.** *Let X be a minimal complex surface of general type and $f : X \to X_{can}$ be the holomorphic map from X to its canonical model contracting the (-2)-curves. Then the Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric ω_∞ as in Theorem 3.2 for any initial Kähler-metric. Furthermore*

$$\omega_\infty = f^*\omega_{KE},$$

where ω_{KE} is the unique smooth Kähler-Einstein orbiford metric on X_{can}. In particular, ω_∞ has local continuous potential, i.e.,

$$\omega_\infty = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$$

for some $\varphi_\infty \in C^0(X)$.

The critical equation for the Monge-Ampère flow on X is given by

$$\frac{(\chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty)^2}{\Omega} = e^{\varphi_\infty}.$$

The following is an immediate corollary from Theorem 3.3 by the fact that $\omega_\infty^2 = \Omega e^{\varphi_\infty}$ and φ_∞ is continuous.

Corollary 3.1. *The Kähler-Einstein volume form on X defined by $\Omega_{KE} = \omega_\infty^2$ is a continuous and nonvanishing volume form on X such that*

$$\sqrt{-1}\partial\overline{\partial}\log\Omega_{KE} = \omega_\infty.$$

4. $\mathrm{kod}(X) = 1$

An elliptic fibration of a surface X is a proper, connected holomorphic map $f : X \to \Sigma$ from X to a curve Σ such that the general fibre is a non-singular elliptic curve. An elliptic surface is a surface admitting an elliptic fibration. Any surface X of $\mathrm{kod}(X) = 1$ must be an elliptic surface. Such an elliptic surface is sometimes called a properly elliptic surface. Since we assume that X is minimal, all fibres are free of (-1)-curves. A very simple example is the product of two curves, one elliptic and the other of genus ≥ 2.

Let $f : X \to \Sigma$ be an elliptic surface. The differential *df* can be viewed as an injection of sheaves $f^*(K_\Sigma) \to \Omega^1_X$. Its cokernel $\Omega_{X/\Sigma}$ is called the sheaf of relative differentials. In general, $\Omega_{X/\Sigma}$ is far from being locally free. If some fibre has a multiple component, then *df* vanishes along this component and $\Omega_{X/\Sigma}$ contains a torsion subsheaf with one-dimensional support. Away from the singularities of f we have the following exact sequence

$$0 \to f^*(K_\Sigma) \to \Omega^1_X \to \Omega_{X/\Sigma} \to 0$$

including an isomorphism between $\Omega_{X/\Sigma}$ and $K_X \otimes f^*(K_\Sigma^\vee)$. We also call the line bundle $\Omega_{X/\Sigma}$ the dualizing sheaf of f on X. The following theorem is well-known (c.f. [BHPV]).

Theorem 4.1 (Kodaira)**.** *Let $f : X \to \Sigma$ be a minimal elliptic surface such that its multiple fibres are $X_{s_1} = m_1F_1$, .., $X_{s_k} = m_kF_k$. Then*

$$K_X = f^*(K_\Sigma \otimes (f_{*1}\mathcal{O}_X)^\vee) \otimes \mathcal{O}_X(\sum(m_i - 1)F_i), \tag{4.1}$$

or

$$K_X = f^*(L \otimes \mathcal{O}_X(\sum(m_i - 1)F_i),$$

where L is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_\Sigma)$ on Σ.

Note that $\deg(f_{*1}\mathcal{O}_X)^\vee = \deg(f_*\Omega_{X/\Sigma}) \geq 0$ and the equality holds if and only if f is locally trivial. The following invariant

$$\delta(f) = \chi(\mathcal{O}_X) + \left(2g(\Sigma) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i})\right)$$

determines the Kodaira dimension of X.

Proposition 4.1. (cf. [BHPV]) *Let $f : X \to \Sigma$ be a relatively minimal elliptic fibration and X be compact. Then* $\mathrm{kod}(X) = 1$ *if and only if* $\delta(f) > 0$.

Kodaira classified all possible singular fibres for f. A fibre X_s is stable if

(1) X_s is reduced,
(2) X_s contains no (-1)-curves,
(3) X_s has only node singularities.

The only stable singular fibres are of type I_b for $b > 0$, therefore such singular fibres are particularly interesting. Let $\mathcal{S}_1 = \{z \in \mathbf{C} \mid \mathrm{Im} z > 0\}$ be the upper half plane and $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})/\{\pm 1\}$ be the modular group acting by $z \to \frac{az+b}{cz+d}$. Then $\mathcal{S}_1/\Gamma_1 \cong \mathbf{C}$ is the period domain or the moduli space of elliptic curves. The j-function gives an isomorphism $\mathcal{S}_1/\Gamma_1 \to \mathbf{C}$ with

(1) $j(z) = 0$ if $z = e^{\frac{\pi}{3}\sqrt{-1}}$ modular Γ_1,
(2) $j(z) = 1$ if $z = \sqrt{-1}$ modular Γ_1.

Let $\Sigma_{reg} = \{s \in \Sigma \mid X_s \text{ is a nonsingular fibre}\}$ Then any elliptic surface $f : X \to \Sigma$ gives a period map $p : \Sigma_{reg} \to \mathcal{S}_1/\Gamma_1$. Set $J : \Sigma_{reg} \mapsto \mathbf{C}$ by $J(s) = j(p(s))$. For a stable fibre X_s of type I_b, the functional invariant J has a pole of order b at s and the monodromy is given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Again we apply the normalized Kähler-Ricci flow (3.1) on such elliptic surfaces of Kodaira dimension one. We choose a semi-positive $(1,1)-$form $\chi \in -c_1(X)$ to be the pullback of a Kähler form on the base Σ and χ might vanish somewhere due to the presence of singular fibres. Let Ω be the smooth volume form on X such that $\sqrt{-1}\partial\overline{\partial} \log \Omega = \chi$. The Kähler class of ω deformed by the Kähler-Ricci flow is given by

$$[\omega] = (1 - e^{-t})[\chi] + e^{-t}[\omega_0],$$

with the initial Kähler metric ω_0. So we let $\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0$ be the reference metric and $\omega = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi$.

As we discussed in the previous section, one can reduce the Kähler-Ricci flow (2.2) to the following Monge-Ampère flow on Kähler potentials.

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{e^{-t}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \tag{4.2}$$

The simple example is the Kähler-Ricci flow on $X = E \times C$, where E is an elliptic curve and C is a curve of genus greater than one. Let $\pi_1 : X \to E$ and $\pi_2 : X \to C$ be the projection maps. Let ω_E be a flat metric on E, ω_C be the hyperbolic metric on C with $\omega_C \in -c_1(C)$ and

$$\omega_0 = \pi_1^*\omega_E + \pi_2^*\omega_C$$

be the initial Kähler metric on X for the Kähler-Ricci flow. Then

$$\omega_t = \pi_2^*\omega_C + e^{-t}\pi_1^*\omega_E$$

solves the Kähler-Ricci flow by

$$\frac{\partial \omega_t}{\partial t} = -e^{-t}\pi_1^*\omega_E = -Ric(\omega_t) - \omega_t.$$

And

$$\lim_{t\to\infty} \omega_t = \pi_2^*\omega_C.$$

The following general result is proved in [SoTi]

Theorem 4.2 (Song-Tian)**.** *Let $f : X \to \Sigma$ be a minimal elliptic surface of* $\mathrm{kod}(X) = 1$. *Let $\Sigma_{reg} = \Sigma \setminus \{s_1, \ldots, s_k\}$. Then the Kähler-Ricci flow has a global solution $\omega(t, \cdot)$ for any initial Kähler metric satisfying:*

(1) *$\omega(t,\cdot)$ converges to $f^*\omega_\infty \in -c_1(X)$ as currents for a positive current ω_∞ on Σ,*
(2) *ω_∞ is smooth on Σ_{reg} and satisfies as currents on Σ*

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{WP} + \sum_{i=1}^{k} \frac{m_i - 1}{m_i}[s_i],$$

(3) *for any compact subset $K \in f^{-1}(\Sigma_{reg})$, there is a constant C_K such that*

$$||\omega||_{L^\infty(K)} + \sup_{s\in f(K)} ||e^t\omega|_{f^{-1}(s)}||_{L^\infty(f^{-1}(s))} + ||S||_{L^\infty(K)} \leq C_K,$$

where S is the scalar curvature of $\omega(t,\cdot)$.

The Kähler-Ricci flow collapses exponentially fast along the vertical direction and the limit metric satisfies a generalized Kähler-Einstein equation (2) with the correction term as the curvature of the fibration. The generalized Kähler-Einstein equation can also be considered as a local version of Kodaira's adjunction formula (4.1).

Corollary 4.1. *Let $f : X \to \Sigma$ be an elliptic fibre bundle over a curve Σ of genus greater than one. Then the Kähler-Ricci flow (2.2) has a global solution with any initial Kähler metric. Furthermore, $\omega(t,\cdot)$ converges with uniformly bounded scalar curvature to the pullback of the Kähler-Einstein metric on Σ.*

If $f : X \to \Sigma$ has only singular fibres of type mI_0, all the smooth fibres of f are isomorphic so that the period map p is trivial with its image as a point in $\mathcal{M}$. The Kähler-Ricci flow will then converge to a cone hyperbolic metric on Σ and the cone singularities appear from where the multiple fibres sit.

5. kod$(X) = 0$

If X is a minimal Kähler surface of kod$(X) = 0$, the canonical line bundle K_X is numerically trivial so that $c_1(X) = 0$. Yau's solution to Calabi's conjecture shows that there always exists a Ricci-flat Kähler metric in any given Kähler class on X. An alternative proof is given by Cao [Ca] by applying the following Kähler-Ricci flow,

$$\begin{cases} \frac{\partial\omega}{\partial t} = -Ric(\omega) \\ \omega\,|_{t=0} = \omega_0 \end{cases} \tag{5.1}$$

where ω_0 is a Kähler metric on X.

Theorem 5.1 (Cao). *Let X be a minimal complex surface with $c_1(X) = 0$. Then the Kähler-Ricci flow (5.1) converges for any initial Kähler metric ω_0 to the unique Ricci-flat metric $\omega_{KE} \in [\omega_0]$ with*

$$Ric(\omega_{KE}) = 0.$$

6. Fano surfaces

A compact complex surface X with $c_1(X) > 0$ is called a Fano surface. The anti-canonical bundle K_X^{-1} is thus positive. By the Enriques-Kodaira classification, $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2 \# n\overline{\mathbf{CP}^2}$ for $0 \leq n \leq 8$ are the only compact differential four-manifolds on which there is a complex structure with positive first Chern class. Tian [Ti2] proves the following theorem for the existence of a Kähler-Einstein metric on Fano surfaces.

Theorem 6.1 (Tian)**.** *Any compact complex surface X with $c_1(X) > 0$ admits a Kähler-Einstein metric if the Lie algebra of the automorphism group on X is reductive.*

So there always exists a Kähler-Einstein metric on $\mathbf{CP}^2$, $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2 \# n\overline{\mathbf{CP}^2}$ for $3 \leq n \leq 8$.

However, there does not exist any Kähler-Einstein metric neither on $\mathbf{CP}^2 \# 1\overline{\mathbf{CP}^2}$ nor on $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}^2}$ since the Lie algebra of the automorphism group on these toric Fano surfaces is not reductive. One can also disprove the existence of any Kähler-Einstein metric by showing the Futaki invariant on these two surfaces is non-zero.

Definition 6.1. *Let X be a complex surface with $c_1(X) > 0$. A Kähler metric ω is called a Kähler-Ricci soliton if it satisfies*

$$Ric(\omega) = \omega + L_V \omega, \tag{6.1}$$

where V is a holomorphic vector field on X and L_V denotes the Lie derivative along V.

A Kähler-Einstein metric is also a special Kähler-Ricci soliton by taking $V = 0$. In [Koi], Koiso constructed a Kähler-Ricci soliton on $\mathbf{CP}^2 \# 1\overline{\mathbf{CP}^2}$. A general result on the existence of Kähler-Einstein metrics and Kähler-Ricci soliton on toric manifolds is proved in [WaZh].

Theorem 6.2 (Wang-Zhu)**.** *There exists a Kähler-Ricci soliton on a toric Kähler manifold with positive first Chern class.*

Therefore there always exists a Kähler-Einstein metric or a Kähler-Ricci soliton on a complex surface of positive first Chern class.

We apply the the following normalized Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega) + \omega \\ \omega\,|_{t=0} = \omega_0 \end{cases} \tag{6.2}$$

with the initial Kähler metric $\omega_0 \in c_1(X)$ so that the Kähler class $[\omega]$ will not change.

Perelman proves a gradient estimate on the Kähler-Ricci flow (6.2), which implies the scalar curvature will stay uniformly bounded (see [SeTi]). He has also claimed that the Kähler-Ricci flow converges (6.2) on a Kähler-Einstein manifold X with positive first Chern class to a Kähler-Einstein metric. This is proved by Tian and Zhu [TiZhu] and generalized to the Kähler-Ricci solitons. Suppose X admits a Kähler-Ricci soliton with respect to the holomorphic vector field V. Then the imaginary part of V induces a one-parameter subgroup G_V in the automorphism group of X.

Theorem 6.3 (Tian-Zhu). *Let X be a compact complex surface of $c_1(X) > 0$. Then the Kähler-Ricci flow will converge to a Kähler-Einstein metric or a Kähler-Ricci soliton if the initial Kähler metric is G_V-invariant.*

7. Generalizations

The generalized Kähler-Einstein metric on the canonical model of an elliptic surface with Kodaira dimension one can be generalized and defined on a family of surfaces with a fibration structure.

Mirror symmetry and the SYZ conjecture make predictions for Calabi-Yau manifolds with "large complex structure limit point" (cf. [StYaZa]). It is believed that in the large complex structure limit, the Ricci-flat metrics should converge in the Gromov-Hausdorff sense to a half-dimensional sphere by collapsing a special Lagrangian torus fibration over this sphere. This holds trivially for elliptic curves and is proved by Gross and Wilson (cf. [GrWi]) in the case of $K3$ surfaces. The method of the proof is to find a good approximation for the Ricci-flat metrics near the large complex structure limit. The approximation metric is obtained by gluing together the Oogrui-Vafa metrics near the singular fibres and a semi-flat metric on the regular part of the fibration. Such a limit metric of $K3$ surfaces is McLean's metric.

We will apply a deformation for a family of Calabi-Yau metrics and derive McLean's metric [Mc] without writing down an accurate approximation metric. Such a deformation can be also done in higher dimensions. It will be interesting to have a flow which achieves this limit. The large complex structure limit of a $K3$ surface $\hat{X}$ can be identified as the mirror to the large Kähler limit of X as shown in [GrWi], so we can fix the complex structure on X and deform the Kähler class to infinity. Let $f : X \to \mathbf{CP}^1$ be an elliptic $K3$ surface. Let $\chi \geq 0$ be the pullback of a Kähler form on $\mathbf{CP}^1$ and ω_0 be a Kähler form on X. We construct a reference Kähler metric $\omega_t = \chi + e^{-t}\omega_0$ and $[\omega_t]$ tends to $[\chi]$ as $t \to \infty$. We can always scale ω_0 so that the volume of each fibre of f with respect to ω_t is e^{-t}. Suppose Ω is a Ricci-flat volume form on X with $\partial\overline{\partial}\log\Omega = 0$. Then Yau's proof [Ya1] of Calabi's conjecture yields a unique solution φ_t to the following Monge-Ampère equation for $t \in [0, \infty)$

$$\begin{cases} \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^2}{\Omega} = C_t \\ \int_X \varphi_t \Omega = 0, \end{cases} \tag{7.1}$$

where $C_t = [\omega_t]^2$. Therefore we obtain a family of Ricci-flat metrics $\omega(t,\cdot) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$. The following theorem is proved in [SoTi].

Theorem 7.1. *Let $f : X \to \mathbf{CP}^1$ be an elliptically fibred $K3$ surface with 24 singular fibres of type I_1. Then the Ricci-flat metrics $\omega(t,\cdot)$ converges to the pullback of a Kähler metric ω_∞ on $\mathbf{CP}^1$ in any compact set of X_{reg} in $C^{1,1}$ as $t \to \infty$. The Kähler metric ω_∞ on $\mathbf{CP}^1$ satisfies the equation*

$$Ric(\omega_\infty) = \omega_{WP}. \tag{7.2}$$

This limit metric ω_∞ coincides with McLean's metric as obtained by Gross and Wilson [GrWi]. Their construction is certainly much more delicate and gives an accurate approximation near the singular fibres by the Ooguri-Vafa metrics. Also McLean's metric is an example of the generalized Kähler-Einstein metric defined as

$$Ric(\omega) = -\lambda\omega + \omega_{WP}$$

when $\lambda = 0$.

In fact, these canonical metrics belong to a class of Kähler metrics defined in [SoTi], which generalize Calabi's extremal metrics. Let Y be a Kähler manifold of complex dimension n together with a fixed closed (1,1)-form θ. Fix a Kähler class $[\omega]$, denote by $\mathcal{K}_{[\omega]}$ the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form $\omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi$. One may consider the following equation

$$\bar\partial V_\varphi = 0, \tag{7.3}$$

where V_φ is defined by

$$\omega_\varphi(V_\varphi, \cdot) = \bar\partial(S(\omega_\varphi) - tr_{\omega_\varphi}(\theta)). \tag{7.4}$$

Clearly, when $\theta = 0$, (7.3) is exactly the equation for Calabi's extremal metrics. For this reason, we call a solution of (7.3) a generalized extremal metric. If Y does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric ω_φ satisfies

$$S(\omega_\varphi) - tr_{\omega_\varphi}(\theta) = \mu, \tag{7.5}$$

where μ is the constant given by

$$\mu = \frac{n(c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $c_1(Y) - [\theta] = \lambda[\omega]$, then any such a metric satisfies

$$Ric(\omega_\varphi) = \lambda\omega_\varphi + \theta,$$

that is, ω_φ is a generalized Kähler-Einstein metric.

Another example of such extremal metrics is constructed by Fine [Fi]. Let $f : X \to \Sigma$ be a Kähler surface admitting a non-singular holomorphic fibration over Σ, with fibres of genus at least 2. We also assume $c_1(\Sigma) \leq 0$. Let V be the vertical tangent bundle of X and $[\omega_t] = -f^*c_1(\Sigma) - e^{-t}c_1(V)$.

Let χ be a Kähler form in $-c_1(\Sigma)$ and $\bar\omega \in -c_1(V)$. Then $\bar\omega = \omega_H \oplus \theta\chi$, where ω_H is the hyperbolic Kähler form on each fiber and θ is a smooth function on X. We then set

$$\omega_t = \chi + e^{-t}\bar\omega.$$

The following theorem is proved by Fine in [Fi].

Theorem 7.2. *For sufficiently large $t \geq 0$, there exists a constant scalar curvature Kähler metric in $[\omega_t]$. Furthermore, such a family of metrics converge to a Kähler metric ω_∞ on Σ defined by*

$$S(\omega_\infty) = tr_{\omega_\infty}(\omega_{WP}) + const, \tag{7.6}$$

where ω_{WP} is the pullback of the Weil-Petersson metric from the moduli spaces of the fibre curves with a certain polarization.

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